Lecture IV: Expansion history and distance measures in cosmology

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I. INTRODUCTION

Having considered the Friedmann equations, let's now turn to the expansion history of the Universe. In this lecture, we'll study phenomenologically how the expansion of the universe relates to the matter content. We will then study how distances in cosmology are measured. In an expanding universe, even the concept of "distance" needs to be treated carefully.

We'll focus on the "pure theory" here, and then in the next lecture look at the observations that have been used to measure H_0, $\Omega_{\rm m}$, and Ω_Λ. Our objective will be to compute, for a given cosmological model, $D(z)$, i.e. the relationship between the distance to an object and the redshift.

II. DENSITY PARAMETERS

Last time, we derived the *critical density* of the Universe, $\rho_{\rm cr}$. This can be written as

$$\rho_{\rm cr} = \frac{3H_0^2}{8\pi G} = 1.88\times10^{-29}h^2\,{\rm g/cm^3}, \tag{1}$$

where we have defined the reduced Hubble parameter by

$$h = \frac{H_0}{100\,{\rm km/s/Mpc}} \approx 0.7. \tag{2}$$

(The use of h is historical, and was introduced to cosmology before the value of H_0 was well determined. We are now stuck with it.)

The density of the Universe can be related to the critical density via the ratio

$$\Omega = \frac{\rho}{\rho_{\rm cr}}; \tag{3}$$

the Friedmann equation then tells us that the Universe is open, flat, or closed if $\Omega < 1$, $\Omega = 1$, or $\Omega > 1$ respectively. Normally Ω is quoted today; the subscript $_0$ or dependence (z) will be used for emphasis if we need to be more general. If the Universe contains matter, radiation, a cosmological constant (Λ: constant density), or anything else, then their densities and hence density parameters add:

$$\Omega = \Omega_{\rm m} + \Omega_{\rm r} + \Omega_\Lambda + \Omega_{\rm other}. \tag{4}$$

Dividing the Friedmann equation by H^2, we see that

$$H^2 + \frac{K}{a^2} = \frac{8}{3}\pi G\rho \qquad \to \qquad 1 + \frac{K}{a^2H^2} = \frac{\rho}{\rho_{\rm cr}} = \Omega. \tag{5}$$

It is thus common practice to supplement this by defining the curvature parameter:

$$\Omega_K = 1 - \Omega = -\frac{K}{a^2H^2}, \tag{6}$$

which is positive (open), zero (flat), or negative (closed) depending on the spatial curvature. It is important to remember that Ω_K does *not* represent a physical contribution to the density of the Universe – it represents spatial curvature. It might help to think of $\Omega_K\rho_{\rm cr}$ as the extra amount of energy density that would be required to flatten the Universe. The curvature can be written in terms of Ω_K as

$$K = -\Omega_K H_0^2, \tag{7}$$

where once again we quote Ω_K today.

A. Examples

The density of any component of the universe today can be related to the density parameter and Hubble rate by

$$\rho_X = \rho_{\rm cr}\Omega_X = 1.88 \times 10^{-29}\Omega_X h^2 \text{ g/cm}^3. \tag{8}$$

For this reason, you will often see densities in cosmology quoted as "$\Omega_X h^2$" rather than in g/cm^3, but understand that this carries the same information. Also note that the densities sum to the total in the sense that $\sum_X \Omega_X h^2 = \Omega h^2$.

Let's start with the cosmic microwave background, which has density (in mass units)

$$\rho_{\rm CMB} = \frac{a_{\rm rad}T^4}{c^2} = 4.64 \times 10^{-34} \text{ g/cm}^3 \tag{9}$$

at $T = 2.725$ K. We can thus see that

$$\Omega_{\rm CMB}h^2 = 2.47 \times 10^{-5}. \tag{10}$$

Evidently, if h is of order unity, the CMB does not have nearly enough energy density to explain the flatness of the universe.

We'll learn later that there is a cosmic neutrino background, which is also a thermal relic from the Big Bang. If neutrinos were massless, their energy density would be of the same order of magnitude as the photons (we'll compute the numerical factor later); combined, the photons and neutrinos give the total radiation $\Omega_{\rm r}$. We now know that neutrinos have mass, and so at least some of them are non-relativistic today. However, at early times when their contribution to the cosmic energy budget is significant, we would have had $k_{\rm B}T > m_\nu c^2$, and so the neutrinos could be treated to a first approximation as massless particles. Neutrino masses are expected to have percent-level impacts on cosmological observables, since their energy density today is not truly negligible, and measuring this is a major goal of next-generation observational projects.

The *Planck* satellite measurements of the densities of baryonic (normal) matter and dark matter are

$$\Omega_{\rm b}h^2 = 0.02230 \pm 0.00014 \quad \text{and} \Omega_{\rm DM}h^2 = 0.1188 \pm 0.0010; \tag{11}$$

the total of these is the matter density, $\Omega_{\rm m}h^2 = 0.14170 \pm 0.00097$. The method of measuring these will be described later, after we consider cosmological perturbations and CMB anisotropies.

III. DENSITY EVOLUTION AND EXPANSION HISTORY

Let's consider a set of constituents of the Universe X (where X may be matter, radiation, Λ, ...) each with a constant equation of state w_X. Then

$$\rho_X = \rho_{X0}a^{-3(1+w_X)} = \rho_{\rm cr}\Omega_X a^{-3(1+w_X)}, \tag{12}$$

and the total density of the Universe varies with scale factor as

$$\rho(a) = \sum_X \rho_X(a) = \rho_{\rm cr}\sum_X \Omega_X a^{-3(1+w_X)}. \tag{13}$$

The Friedmann equation then says

$$H^2 + \frac{K}{a^2} = \frac{8}{3}\pi G\rho = \frac{8}{3}\pi G\rho_{\rm cr}\sum_X \Omega_X a^{-3(1+w_X)} = H_0^2\sum_X \Omega_X a^{-3(1+w_X)}. \tag{14}$$

Using Eq. (7) and moving K to the right-hand side, we get:

$$H^2 = H_0^2\left[\frac{\Omega_K}{a^2} + \sum_X \Omega_X a^{-3(1+w_X)}\right]. \tag{15}$$

It is common to define the energy function:

$$\mathcal{E}(a) = \frac{\Omega_K}{a^2} + \sum_X \Omega_X a^{-3(1+w_X)}, \tag{16}$$

so that

$$H(a) = H_0\sqrt{\mathcal{E}(a)}. \tag{17}$$

[*Aside*: You can easily see from the above derivation that if w_X is not constant, then in the energy function we should make the replacement

$$a^{-3(1+w_X)} \to \exp\left\{-3\int_1^a [1+w_X(a)]\frac{da}{a}\right\}. \tag{18}$$

We will have only a few occasions to use this form.]

If we can find the function $\mathcal{E}(a)$, we can do the integral to find the time:

$$t = \int dt = \int \frac{da}{\dot{a}} = \int \frac{da}{aH} = \frac{1}{H_0}\int \frac{da}{a\sqrt{\mathcal{E}(a)}} = \frac{-1}{H_0}\int \frac{dz}{(1+z)\sqrt{\mathcal{E}(z)}}. \tag{19}$$

(Recall that we set $t = 0$ at $a = 0$ or $z = \infty$.) A similar rule holds for η:

$$\eta = \int \frac{dt}{a} = \int \frac{da}{a\dot{a}} = \int \frac{da}{a^2 H} = \frac{1}{H_0}\int \frac{da}{a^2\sqrt{\mathcal{E}(a)}} = \frac{-1}{H_0}\int \frac{dz}{\sqrt{\mathcal{E}(z)}}. \tag{20}$$

We will make extensive use of both Eqs. (19) and (20).

A. Examples

To continue, let's consider the equation of state of each of the constituents described thus far. We know that nonrelativistic matter (both normal and dark) has $w \approx 0$, so $\rho \propto a^{-3}$ (density scales as the inverse of volume). Radiation has $w = \frac{1}{3}$ and $\rho \propto a^{-4}$ (there's an additional factor of a^{-1}, which makes sense since a photon's wavelength λ is stretched by the expansion of the Universe and the energy per photon is h/λ). If there is a **cosmological constant** (denoted by Λ) – a fundamental energy density associated with empty space – then its density must be constant and it must have $w_\Lambda = -1$. We'll explore the physics of the cosmological constant later, but for now we need to include it. In this case, the energy function becomes

$$\mathcal{E}(a) = \Omega_\Lambda + \frac{\Omega_K}{a^2} + \frac{\Omega_{\rm m}}{a^3} + \frac{\Omega_{\rm r}}{a^4} \tag{21}$$

or – in terms of the redshift –

$$\mathcal{E}(z) = \Omega_\Lambda + \Omega_K(1+z)^2 + \Omega_{\rm m}(1+z)^3 + \Omega_{\rm r}(1+z)^4. \tag{22}$$

We can see that the energy function is generally a quartic function of z, and so t and η can be described with elliptic integrals. It is rare for this to be useful (or enlightening) and numerical approaches are usually used. However, in a few limiting cases where only some of the constituents are relevant, we can solve for t and a. These cases are interesting because Λ, K, m, and r have different power-law dependences on a, and usually not more than one or two are important at any given time.

1. *Matter and curvature; no Λ*

Prior to the discovery of cosmic acceleration, this was believed to be the case of interest to late-time cosmology (i.e. after the radiation had ceased to be negligible). Here we have only matter and curvature, and $\Omega_K = 1 - \Omega_{\rm m}$. This universe is open, flat, or closed if $\Omega_{\rm m} < 1$, $= 1$, or > 1.

There is no analytic function for $a(t)$ in this case, but we can find a parametric solution. Let's consider the η integral:

$$\eta = \frac{1}{H_0}\int \frac{da}{a^2\sqrt{\Omega_K a^{-2} + \Omega_{\rm m}a^{-3}}} = \frac{1}{H_0}\int \frac{da}{\sqrt{a(\Omega_K a + \Omega_{\rm m})}}. \tag{23}$$

Looking at the right-hand side, we can see that if $\Omega_K \geq 0$ ($\Omega_\mathrm{m} \leq 1$), the integrand is always well-behaved and the expansion of the universe continues forever. If $\Omega_K < 0$, there is a maximum scale factor

$$a_\mathrm{max} = -\frac{\Omega_\mathrm{m}}{\Omega_K} = \frac{\Omega_\mathrm{m}}{\Omega_\mathrm{m} - 1} \tag{24}$$

at which $\mathcal{E}(a) \to 0$, i.e. the expansion of the universe stops. The subsequent evolution is that the universe contracts ($H < 0$, still $a < a_\mathrm{max}$) to a Big Crunch. The collapse is the time-reverse of the expansion (at least as far as the background cosmology is concerned; stars and galaxies of course continue to evolve with the usual arrow of time).

To solve for η, let's first look at the case of $\Omega_K < 0$ (we'll solve the others by analytic continuation). Define the change of variables

$$\xi = \frac{2a}{a_\mathrm{max}} - 1 \qquad \leftrightarrow \qquad a = \frac{1}{2} a_\mathrm{max}(1 + \xi). \tag{25}$$

This substitution re-scales the argument of the square root to move the singularities from $a = 0$ and $a = a_\mathrm{max}$ to $\xi = -1$ and $\xi = 1$:

$$\eta = \frac{1}{(-\Omega_K)^{1/2} H_0} \int \frac{d\xi}{\sqrt{1 - \xi^2}} = \frac{1}{(-\Omega_K)^{1/2} H_0} (\arcsin \xi + C), \tag{26}$$

where the constant of integration C must be $\pi/2$ so that $\eta = 0$ when $\xi = -1$ ($a = 0$). Solving for ξ gives

$$\xi = \sin\left(-\frac{\pi}{2} + (-\Omega_K)^{1/2} H_0 \eta\right) = -\cos[(-\Omega_K)^{1/2} H_0 \eta]. \tag{27}$$

Then

$$a = \frac{\Omega_\mathrm{m}}{2(-\Omega_K)} \{1 - \cos[(-\Omega_K)^{1/2} H_0 \eta]\}. \tag{28}$$

The equation for t can be solved parametrically by integration:

$$t = \int a\, d\eta = \frac{\Omega_\mathrm{m}}{2(-\Omega_K)} \int \{1 - \cos[(-\Omega_K)^{1/2} H_0 \eta]\}\, d\eta = \frac{\Omega_\mathrm{m}}{2(-\Omega_K)} \left\{\eta - \frac{\sin[(-\Omega_K)^{1/2} H_0 \eta]}{(-\Omega_K)^{1/2} H_0}\right\}. \tag{29}$$

The solution to $a(t)$ is thus a *cycloid* – the path traced by a point on a rolling wheel of radius $(-\Omega_K)^{1/2} H_0$.

This model for the Universe ends at conformal time

$$\eta_\mathrm{end} = \frac{2\pi}{(-\Omega_K)^{1/2} H_0} = \frac{2\pi}{\sqrt{K}} = 2\pi R, \tag{30}$$

when a returns to zero. Recall that in conformal time $d\eta$, light travels a comoving distance $d\eta$, so the total comoving distance travelled is $2\pi R$. Thus the closed universe lives exactly long enough for light to carry out a great circle trajectory once. The total lifetime of the Universe in physical time is

$$t_\mathrm{end} = \frac{\Omega_\mathrm{m}}{2(-\Omega_K)} \frac{2\pi}{(-\Omega_K)^{1/2} H_0} = \frac{\Omega_\mathrm{m}}{(-\Omega_K)^{3/2} H_0} = \frac{\Omega_\mathrm{m}}{(\Omega_\mathrm{m} - 1)^{3/2} H_0}. \tag{31}$$

Since $1/H_0$ is currently 14 Gyr, we never thought there was any danger of the Big Crunch as an imminent ecological catastrophe.

We can understand the open model ($\Omega_\mathrm{m} < 1$, $\Omega_K > 0$) by analytically continuing the above expressions:

$$a = \frac{\Omega_\mathrm{m}}{2\Omega_K} \{\cosh[(-\Omega_K)^{1/2} H_0 \eta] - 1\} \quad \text{and} \quad t = \frac{\Omega_\mathrm{m}}{2\Omega_K} \left\{\frac{\sinh[\Omega_K^{1/2} H_0 \eta]}{\Omega_K^{1/2} H_0} - \eta\right\}. \tag{32}$$

The case of $\Omega_\mathrm{m} = 1$ and $\Omega_K = 0$ is left for the homework; the key result is that $a \propto t^{2/3}$. This can be shown either directly by integrating $\mathcal{E}(a)$, or by applying l'Hôpital's rule.

2. Matter and Λ; flat

This is the case that appears to be relevant to our Universe: $\Omega_K = 0$, but with $\Omega_{\rm m} + \Omega_\Lambda = 1$ (at present, we have roughly $\Omega_{\rm m} = 0.3$ and $\Omega_\Lambda = 0.7$). We'll focus here on the range $0 < \Omega_{\rm m} < 1$ as this is the case that appears to be relevant.

The time and conformal time integrals are:

$$t = \frac{1}{H_0}\int \frac{da}{a\sqrt{\Omega_\Lambda + \Omega_{\rm m}a^{-3}}} \tag{33}$$

and

$$\eta = \frac{1}{H_0}\int \frac{da}{a^2\sqrt{\Omega_\Lambda + \Omega_{\rm m}a^{-3}}}. \tag{34}$$

It is seen that as $a \rightarrow 0^+$, the integrands scale as $a^{1/2}\,da$ and $a^{-1/2}\,da$ respectively, so both t and η approach constants (traditionally set to zero) at the Big Bang. On the other hand, as $a \rightarrow \infty$, the t and η integrals go to $a^{-1}\,da$ and $a^{-2}\,da$ respectively. That means that while

$$\lim_{a\rightarrow\infty} t = \infty, \tag{35}$$

we have

$$\lim_{a\rightarrow\infty} \eta = \text{constant} = \eta_{\rm max}. \tag{36}$$

The universe is infinitely long-lived in physical time, but has a finite conformal lifetime, meaning that there is a finite comoving distance that light can travel from the Big Bang to the infinite future. No matter how long we wait, we will *never* be able to see galaxies at comoving distance farther away than $\eta_{\rm max}$. By doing the definite integral, you should be able to show that this conformal lifetime is

$$\eta_{\rm max} = \frac{\Gamma(1/3)\Gamma(1/6)}{3\sqrt{\pi}\,\Omega_\Lambda^{2/3}\Omega_{\rm m}^{1/3}H_0} \approx \frac{2.8044}{\Omega_\Lambda^{2/3}\Omega_{\rm m}^{1/3}H_0}. \tag{37}$$

Because $w_\Lambda = -1$, the cosmological constant has negative pressure, and this contributes negatively to deceleration. From the second Friedmann equation,

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho+3p) = -\frac{4}{3}\pi G(\rho_{\rm m}+\rho_\Lambda-3\rho_\Lambda) = -\frac{4}{3}\pi G\rho_{\rm cr}(\Omega_{\rm m}a^{-3}-2\Omega_\Lambda) = \frac{1}{2}H_0^2(2\Omega_\Lambda-\Omega_{\rm m}a^{-3}). \tag{38}$$

We can thus see that the expansion of the universe decelerates ($\ddot{a} < 0$) until the inflection epoch

$$a_{\rm inflection} = \sqrt[3]{\frac{\Omega_{\rm m}}{2\Omega_\Lambda}} \approx 0.6 \tag{39}$$

and accelerates ($\ddot{a} > 0$) thereafter.

For reference, there exist series expansions in a that are useful at early times (i.e. up to the epoch of matter-Λ equality). We can define the scale factor at matter-Λ equality $a_{{\rm m}\Lambda}$ by

$$a_{{\rm m}\Lambda} = \sqrt[3]{\frac{\Omega_{\rm m}}{\Omega_\Lambda}}, \tag{40}$$

and then see that

$$\begin{aligned} t &= \frac{1}{\Omega_{\rm m}^{1/2}H_0}\int \frac{da}{a\sqrt{a_{{\rm m}\Lambda}^{-3}+a^{-3}}} \\ &= \frac{1}{\Omega_{\rm m}^{1/2}H_0}\int a^{1/2}\left(1+\frac{a^3}{a_{{\rm m}\Lambda}^3}\right)^{-1/2} da \\ &= \frac{2}{3\Omega_{\rm m}^{1/2}H_0}a^{3/2}\left(1-\frac{1}{6}\frac{a^3}{a_{{\rm m}\Lambda}^3}+\frac{3}{40}\frac{a^6}{a_{{\rm m}\Lambda}^6}-\frac{5}{112}\frac{a^9}{a_{{\rm m}\Lambda}^9}+...\right). \end{aligned} \tag{41}$$

Please note, however, that this series doesn't converge at $a > a_{{\rm m}\Lambda}$, so it is not useful today. Numerical integration is greatly preferred.

3. Matter and radiation; flat

A final case, useful in the early universe, is when the cosmological constant and curvature are negligible but we must consider both matter and radiation. We define the **scale factor at equality** by

$$a_{\rm eq} = \frac{\Omega_{\rm r}}{\Omega_{\rm m}}, \tag{42}$$

so that the universe is **radiation-dominated** ($\rho_{\rm r} > \rho_{\rm m}$) at $a < a_{\rm eq}$ and **matter-dominated** ($\rho_{\rm m} > \rho_{\rm r}$) at $a > a_{\rm eq}$.

Writing the new scale parameter $y = a/a_{\rm eq}$ (basically the scale factor relative to equality), we see that

$$t = \frac{1}{H_0}\int \frac{da}{a\sqrt{\Omega_{\rm r}a^{-4} + \Omega_{\rm m}a^{-3}}} = \frac{a_{\rm eq}^2}{\Omega_{\rm r}^{1/2}H_0}\int \frac{y\,dy}{\sqrt{1+y}} = \frac{2a_{\rm eq}^2}{3\Omega_{\rm r}^{1/2}H_0}\left[(y-2)\sqrt{1+y}+2\right] \tag{43}$$

and

$$\eta = \frac{1}{H_0}\int \frac{da}{a^2\sqrt{\Omega_{\rm r}a^{-4} + \Omega_{\rm m}a^{-3}}} = \frac{a_{\rm eq}}{\Omega_{\rm r}^{1/2}H_0}\int \frac{dy}{\sqrt{1+y}} = \frac{2a_{\rm eq}^2}{\Omega_{\rm r}^{1/2}H_0}\left[\sqrt{1+y}-1\right]. \tag{44}$$

IV. THE DISTANCE MEASURES

We've already come up with one measure of distance to an object: the **comoving distance** χ. If we see an object at some redshift z, then we know that light came from that object to us on a curve with $d\chi/d\eta = -1$, and so the comoving distance is simply:

$$\chi = \eta_0 - \eta. \tag{45}$$

Using our equation from Lecture III, that

$$\eta = -\frac{1}{H_0}\int \frac{dz}{\sqrt{\mathcal{E}(z)}}, \tag{46}$$

we see that

$$\chi(z) = \frac{1}{H_0}\int_0^z \frac{dz}{\sqrt{\mathcal{E}(z)}} = \frac{1}{H_0}\int_0^z \frac{dz}{\sqrt{\Omega_\Lambda + \Omega_K(1+z)^2 + \Omega_{\rm m}(1+z)^3}}. \tag{47}$$

For studies of the nearby universe, it is convenient to Taylor-expanding this:

$$\chi(z) = \frac{1}{H_0}\left[z - \frac{2\Omega_K + 3\Omega_{\rm m}}{4}z^2 + \frac{-4\Omega_K - 12\Omega_{\rm m} + 12\Omega_K^2 + 36\Omega_K\Omega_{\rm m} + 27\Omega_{\rm m}^2}{24}z^3 + ...\right]. \tag{48}$$

Note, however, that at z exceeding a few tenths numerical integration is preferred.

A special case of interest that can be done exactly is the **Einstein-de Sitter model** $\Omega_{\rm m} = 1$, $\Omega_\Lambda = \Omega_K = 0$, which leads to

$$\chi(z) = \frac{2}{H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right]. \tag{49}$$

While useful for building intuition and describing some general aspects of cosmology and getting the orders of magnitudes of things, the Einstein-de Sitter model is not a good approximation to the real universe.

This equations are the basis for much of distance determination in cosmology. However, we normally define "distance" through some sort of physical measurement, which may or may not be equivalent to measuring χ. Those measurements will be our next subject.

A. Angular diameter distances

One method of measuring the distance to an object is to measure the angular size $d\theta$ of a feature of known size ds. The **angular diameter distance** is defined as the ratio of physical size to apparent angular size. It is given by the ratio $ds/d\theta$, and corresponds to the physical distance in ordinary Euclidean geometry. The angular diameter distance is given by

$$D_A(z) = \frac{ds}{d\theta} = ar = \frac{r}{1+z}. \tag{50}$$

To determine it, we need to know $r(z)$, which can be obtained from Eq. (47) and the rules:

$$r(z) = \begin{cases} S\sinh(\chi/S) & K<0 \\ \chi & K=0 \\ R\sin(\chi/R) & K>0 \end{cases}. \tag{51}$$

The quantity $r(z)$ is often known as the **comoving angular diameter distance** since it is the ratio of comoving size to apparent angular size.

The Taylor expansion is once again useful; first, we Taylor-expand the relation $r(\chi)$:

$$r(\chi) = \chi - \frac{1}{6}K\chi^3 + ... = \chi + \frac{1}{6}\Omega_K H_0^2\chi^3 + ..., \tag{52}$$

which – on substitution into Eq. (47) – yields

$$r(z) = \frac{1}{H_0}\left[z - \frac{2\Omega_K + 3\Omega_{\rm m}}{4}z^2 + \frac{-12\Omega_{\rm m} + 12\Omega_K^2 + 36\Omega_K\Omega_{\rm m} + 27\Omega_{\rm m}^2}{24}z^3 + ...\right]. \tag{53}$$

The Taylor expansion of the angular diameter distance is thus

$$D_A(z) = \frac{1}{H_0}\left[z - \frac{4 + 2\Omega_K + 3\Omega_{\rm m}}{4}z^2 + \frac{24 + 12\Omega_K + 6\Omega_{\rm m} + 12\Omega_K^2 + 36\Omega_K\Omega_{\rm m} + 27\Omega_{\rm m}^2}{24}z^3 + ...\right]. \tag{54}$$

We can see that if the order-z^2 term is measured, then measurements of the angular diameter distance at low redshift constrain the combination $2\Omega_K + 3\Omega_{\rm m} = 2 - 2\Omega_\Lambda + \Omega_{\rm m}$, and at higher redshift the contributions of Λ and matter can be disentangled.

As one goes to even higher redshift, it is useful to consider to the Einstein-de Sitter model, for which:

$$D_A(z) = \frac{2}{H_0(1+z)}\left[1 - \frac{1}{\sqrt{1+z}}\right]. \tag{55}$$

This passes through a maximum at $z = 1.25$, where $D_A(z) = 0.296/H_0$. At large z, the angular diameter distance goes to zero! This behavior is generic, and is a result of the expansion of the universe: even a physically small object at high redshift can appear very large on the sky.

B. Luminosity distance

The luminosity distance $D_L(z)$ is defined by the relation:

$$F = \frac{L}{4\pi D_L^2}, \tag{56}$$

where L is the intrinsic luminosity (in e.g. W) of an object at redshift z, and F is the apparent flux (in W/m^2).

The luminosity distance can be calculated by considering how light from an object propagates to the observer. Consider the radiation emitted by the object in proper time Δt; this has initial energy $L\,\Delta t$, and occupies a shell of thickness Δt (really $c\,\Delta t$, but $c = 1$). By the present day, the energy of that radiation has declined by a factor of $1+z$, the thickness of the shell has been stretched to $(1+z)\Delta t$, and the packet of radiation is spread over a surface area of $4\pi r^2$. Therefore, the energy per unit area per unit time (i.e. divided by the shell thickness) is

$$F = \frac{\Delta E_{\rm today}}{A\,\Delta t_{\rm today}} = \frac{L\,\Delta t/(1+z)}{4\pi r^2\,(1+z)\Delta t} = \frac{L}{4\pi r^2(1+z)^2}, \tag{57}$$

and so the luminosity distance, defined by Eq. (56), is

$$D_L(z) = (1+z)r(z). \tag{58}$$

The Taylor expansion is

$$D_L(z) = \frac{1}{H_0}\left[z + \frac{4-2\Omega_K-3\Omega_{\rm m}}{4}z^2 + \frac{24-12\Omega_K-6\Omega_{\rm m}+12\Omega_K^2+36\Omega_K\Omega_{\rm m}+27\Omega_{\rm m}^2}{24}z^3+...\right]. \tag{59}$$

We note that we *always* have

$$D_L(z) = (1+z)^2 D_A(z), \tag{60}$$

so these two types of distances carry exactly the same information about cosmology. That is, cosmic distances measured by "standard candles" (objects of known intrinsic luminosity whose flux can be measured) carry the same information as distances measured by "standard rulers" (objects of known intrinsic size whose angular size can be measured).

In the Einstein-de Sitter model, we have

$$D_L(z) = \frac{2}{H_0}\left[1+z-\sqrt{1+z}\right]. \tag{61}$$

In all cases, at high z, the luminosity distance goes to ∞. That is, while at very high redshift an intrinsically small object can appear large, an intrinsically faint object cannot appear bright. This is a sad fact for observations of high-z galaxies.

Real detectors are only sensitive to some portion of the electromagnetic spectrum. In this case, we are interested not in the *total* (or **bolometric**) flux in W/m^2, but the *spectral* flux F_ν (in W/m^2/Hz: the amount of energy reaching a detector per unit time per unit area per unit frequency). Since frequencies redshift by a factor of $1+z$ – i.e. the frequency observed and emitted are related by $\nu_{\rm em} = (1+z)\nu_{\rm obs}$ – the spectral flux is related to the spectral luminosity (L_ν: W/Hz) by

$$F_{\nu_{\rm obs}}\Delta\nu_{\rm obs} = \frac{L_{\nu_{\rm em}}\Delta\nu_{\rm em}}{4\pi D_L^2}. \tag{62}$$

Using that $\Delta\nu_{\rm em} = (1+z)\Delta\nu_{\rm obs}$, we see that

$$F_\nu = \frac{(1+z)L_{(1+z)\nu}}{4\pi D_L^2}. \tag{63}$$

C. Parallax distance

Finally, let's consider distances measured by parallax. We can't measure cosmological distances this way (yet), but the distance measure so constructed also appears in e.g. the study of gravitational lenses.

Let's consider two neighboring observers B and C, looking at a distant object A, with the angle $\angle ACB$ right. Then as seen from B, object A appears displaced by $\frac{\pi}{2}-\beta$ from the position of A as seen by C, where $\beta = \angle ABC$. (See geometry in Fig. 1.) If the side length from B to C is denoted by a, and we take a to be small, then the **parallax distance** is

$$D_\Pi = \lim_{a\to 0}\frac{a}{\pi/2-\beta}. \tag{64}$$

You can compute the parallax distance using the tools on Homework 1. Taking the case of a closed Universe, and working with comoving sides of the triangle a, b, and c, you know that

$$\sin\beta = \frac{\sin(b/R)}{\sin(c/R)} \quad\text{and}\quad \cos\frac{c}{R} = \cos\frac{a}{R}\cos\frac{b}{R}. \tag{65}$$

Taking the latter equation, we see that

$$\sin\frac{c}{R} = \sqrt{1-\cos^2\frac{a}{R}\cos^2\frac{b}{R}} = \sqrt{1-\cos^2\frac{b}{R}+\cos^2\frac{b}{R}\sin^2\frac{a}{R}} = \sqrt{\sin^2\frac{b}{R}\left(1+\sin^2\frac{a}{R}\cot^2\frac{b}{R}\right)}, \tag{66}$$

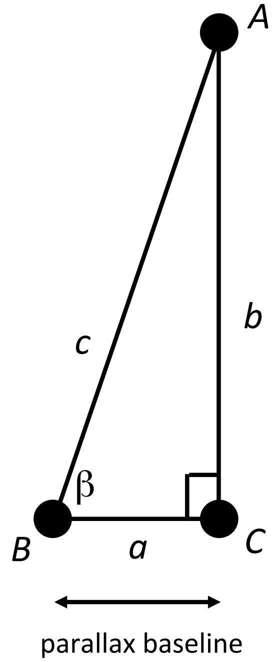

FIG. 1: Geometry for the parallax distance.

and so

$$\sin\beta = \frac{\sin(b/R)}{\sin(c/R)} = \left(1 + \sin^2\frac{a}{R}\cot^2\frac{b}{R}\right)^{-1/2}; \tag{67}$$

using the Pythagorean identities, we learn that

$$\cot\beta = \sin\frac{a}{R}\cot\frac{b}{R}. \tag{68}$$

If we Taylor-expanded in a on the right-hand side (to lowest order, $\sin(a/R) \approx a/R$), and on the left-hand side in β (to lowest order, $\cot\beta \approx \frac{\pi}{2} - \beta$), we see that

$$\frac{\pi}{2} - \beta = \frac{a}{R}\cot\frac{b}{R} + \dots. \tag{69}$$

This leads to the conclusion that in a closed Universe,

$$D_\Pi = R\tan\frac{b}{R} = R\tan\frac{\chi}{R}, \tag{70}$$

where we identify b with the comoving distance to an object at redshift z. Analytic continuation to imaginary R (for open models) and l'Hôpital's rule at $R \to \infty$ (for flat models) allow us to determine the general formula:

$$D_\Pi(z) = \begin{cases} R\tan(\chi/R) & K > 0, \\ \chi & K = 0 \\ S\tanh(\chi/S) & K < 0. \end{cases} \tag{71}$$

Note that in a closed Universe, the parallax distance becomes *infinite* at $\chi = \frac{\pi}{2}R$: an object a quarter-circle away does not change apparent position if the observer moves in a direction perpendicular to the line-of-sight. (Consider looking at the North Pole from the Equator: it is always in the direction north, even if you move in longitude.) Even stranger is that at $\chi > \frac{\pi}{2}R$, the parallax distance becomes negative: an observer moving to the left sees the object appear to move left! Nothing this weird happens in the flat or open universes, where $D_\Pi(z)$ is always positive.

Appendix A: Magnitudes

Since we'll use luminosity distances so much, I've included a cheat sheet here on the use of magnitudes in astronomy. You can find more in any observational astronomy reference, but this section should provide the basics.

The spectral flux of an object F_ν is measured in W/m^2/Hz. In general the photon count rate of the object observed through a telescope that has collecting area A and photon detection efficiency $\eta(\nu)$ can be written as

$$\dot{N} = \int F_\nu \eta(\nu) \frac{A}{h\nu}\, d\nu. \tag{A1}$$

The count rate can be compared to that of a "reference" object with some reference spectrum $F_\nu^{\rm ref}$:

$$\frac{\dot{N}}{\dot{N}^{\rm ref}} = \frac{\int \nu^{-1} F_\nu \eta(\nu)\, d\nu}{\int \nu^{-1} F_\nu^{\rm ref} \eta(\nu)\, d\nu}. \tag{A2}$$

The **apparent magnitude** of the object is this relative count rate, placed on a logarithmic scale with base $10^{0.4} = 2.512$:

$$m = -2.5 \log_{10} \frac{\dot{N}}{\dot{N}^{\rm ref}} = -2.5 \log_{10} \frac{\int \nu^{-1} F_\nu \eta(\nu)\, d\nu}{\int \nu^{-1} F_\nu^{\rm ref} \eta(\nu)\, d\nu}. \tag{A3}$$

Note that the magnitude system is an inverse scale: larger magnitudes refer to fainter objects.

To define the apparent magnitude of an object, we must specify a response $\eta(\nu)$ [or $\eta(\lambda)$] and the reference object, which defines 0th magnitude. The response function of a real telescope is determined by the product of atmospheric transmission (if ground-based), the reflectivity of the mirrors, the transmission of the lenses, the filter (inserted into the telescope to select a range of wavelengths), and the sensitivity of the detector to each wavelength of light. Each telescope and instrument has its own response, although one often quotes a common set of standard filters. Common sets include:

- Optical astronomy: U, B, V, R, and I, centered at 0.36, 0.44, 0.55, 0.66, and 0.81 μm.
- Alternative optical set (SDSS): u, g, r, i, z, centered at 0.35, 0.48, 0.62, 0.76, and 0.90 μm.
- Ground-based near infrared (e.g. UKIRT): Z, Y, J, H, and K, chosen to avoid the regularly spaced H_2O absorption bands in the Earth's atmosphere, and centered at 0.88, 1.03, 1.25, 1.63, and 2.20 μm. [Space-based observatories are not constrained to observe between the H_2O bands, and so may make their own choices.]

Two common choices of reference object are the star Vega (defining the "Vega magnitude" system) and a fictitious reference object with $F_\nu^{\rm ref} = 3.631 \times 10^{-23}$ W/m^2/Hz (defining the "AB magnitude" system). It's generally good to specify which you are using; they are similar in the V band, but in the IR the AB reference can be brighter than Vega by several magnitudes.

For reference, the human eye can see down to $\sim$ 6th magnitude in the visible (V band), most of the sky has been surveyed down to $\sim$ 23rd magnitude, and magnitude 28–29 is possible with long exposures on big telescopes.

The **absolute magnitude** M of an object is the apparent magnitude it would have if it were 10 pc away. The apparent and absolute magnitudes differ by

$$m = M + \mu = M + 5 \log_{10} \frac{D}{10\,{\rm pc}}, \tag{A4}$$

where the correction factor μ is known as the **distance modulus**. It is zero at 10 pc, 5 magnitudes at 100 pc, 10 magnitudes at 1 kpc, etc.