# Physics 6820 – Midterm (with Solutions)

**1. Trajectories in a spherical system.** [16 points]

Let's consider the spacetime:

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2). \tag{1}$$

Depending on the function $f(r)$, this spacetime describes Minkowski spacetime, non-rotating black holes, and de Sitter spacetime (an exponentially expanding Universe). You will investigate the properties of geodesics in this spacetime. Your answers may contain $f$ and derivatives thereof ($f_{,r}$, $f_{,rr}$, etc.).

(a) [4 points] Write down the metric tensor $g_{\mu\nu}$ and the inverse metric $g^{\mu\nu}$ in the $t, r, \theta, \phi$ coordinate system.

This can be done directly from the line element:

$$g_{\mu\nu} \to \begin{pmatrix} -f & 0 & 0 & 0 \\ 0 & 1/f & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \quad g^{\mu\nu} \to \begin{pmatrix} -1/f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1/r^2 & 0 \\ 0 & 0 & 0 & 1/(r^2\sin^2\theta) \end{pmatrix}. \tag{2}$$

(b) [8 points] Write down the non-zero Christoffel symbols $\Gamma^\alpha{}_{\beta\gamma}$ for this metric. (Hint: of the 40 Christoffel symbols that are distinct taking account of symmetry under $\beta \leftrightarrow \gamma$, you should find that 9 are non-zero.)

The non-zero metric derivatives are

$$g_{tt,r} = -f_{,r}, \quad g_{rr,r} = -\frac{f_{,r}}{f^2}, \quad g_{\theta\theta,r} = 2r, \quad g_{\phi\phi,r} = 2r\sin^2\theta, \quad g_{\phi\phi,\theta} = 2r^2\sin\theta\cos\theta. \tag{3}$$

Then the "down" Christoffel symbols are

$$\Gamma_{ttr} = \Gamma_{trt} = -\frac{1}{2}f_{,r}, \quad \Gamma_{rtt} = \frac{1}{2}f_{,r}, \quad \Gamma_{rrr} = -\frac{f_{,r}}{2f^2}, \quad \Gamma_{\theta\theta r} = \Gamma_{\theta r\theta} = r, \quad \Gamma_{r\theta\theta} = -r, \tag{4}$$

$$\Gamma_{\phi\phi r} = \Gamma_{\phi r\phi} = r\sin^2\theta, \quad \Gamma_{r\phi\phi} = -r\sin^2\theta, \quad \Gamma_{\phi\phi\theta} = \Gamma_{\phi\theta\phi} = r^2\sin\theta\cos\theta, \quad \text{and} \quad \Gamma_{\theta\phi\phi} = -r^2\sin\theta\cos\theta. \tag{5}$$

We can now move the first index up:

$$\Gamma^t{}_{tr} = \Gamma^t{}_{rt} = \frac{f_{,r}}{2f}, \quad \Gamma^r{}_{tt} = \frac{1}{2}ff_{,r}, \quad \Gamma^r{}_{rr} = -\frac{f_{,r}}{2f}, \quad \Gamma^\theta{}_{\theta r} = \Gamma^\theta{}_{r\theta} = \frac{1}{r}, \quad \Gamma^r{}_{\theta\theta} = -rf, \tag{6}$$

$$\Gamma^\phi{}_{\phi r} = \Gamma^\phi{}_{r\phi} = \frac{1}{r}, \quad \Gamma^r{}_{\phi\phi} = -rf\sin^2\theta, \quad \Gamma^\phi{}_{\phi\theta} = \Gamma^\phi{}_{\theta\phi} = \frac{\cos\theta}{\sin\theta}, \quad \text{and} \quad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta. \tag{7}$$

(c) [4 points] Write the geodesic equation in this metric for a massive particle, i.e., write equations for $\ddot{t}$, $\ddot{r}$, $\ddot{\theta}$, and $\ddot{\phi}$, where the double-dots are with respect to proper time $\tau$.

We use $\ddot{x}^\alpha = -\Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma$. Then:

$$\ddot{t} = -\frac{f_{,r}}{f}\dot{t}\dot{r}, \quad \ddot{r} = -\frac{1}{2}ff_{,r}\dot{t}^2 + \frac{f_{,r}}{2f}\dot{r}^2 + rf(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2), \quad \ddot{\theta} = -\frac{2}{r}\dot{r}\dot{\theta} + \sin\theta\cos\theta\,\dot{\phi}^2, \quad \ddot{\phi} = -\frac{2}{r}\dot{r}\dot{\phi} - 2\frac{\cos\theta}{\sin\theta}\dot{\theta}\dot{\phi}. \tag{8}$$