# Lecture XXIII: Evolution of the Universe

(Dated: November 22, 2019)

## I. OVERVIEW

We now turn our attention to the overall evolution of the Universe, with an emphasis on the aspects determined by the Friedmann equations.

## II. THE BASIC EQUATIONS

Let's return to the first-order Friedmann equation:

$$3H^2 = 8\pi\rho + \Lambda - \frac{3K}{a^2} \tag{1}$$

and the continuity equation:

$$\dot{\rho} = -3H(\rho + p). \tag{2}$$

We will suppose that the Universe contains constituents $X$ with density $\rho_X$ and pressure $p_X$. We suppose that the *equation of state* of constituent $X$ is $w_X \equiv p_X/\rho_X$. Note that for pressureless matter, $w_X = 0$ and for radiation, $w_X = \frac{1}{3}$. If $w_X$ is constant, then we have

$$\frac{\mathrm{d}\ln\rho_X}{\mathrm{d}\ln a} = \frac{\dot{\rho}_X/\rho_X}{\dot{a}/a} = \frac{-3H(\rho_X + p_X)/\rho_X}{H} = -3(1 + w_X). \tag{3}$$

This implies that a constituent of the Universe with constant equation of state has a density that scales as:

$$\rho_X = \rho_{X0} a^{-3(1+w_X)}. \tag{4}$$

If we add up these densities and plug them into Eq. (1), then we arrive at the equation:

$$3H^2 = 8\pi \sum_X \rho_{X0} a^{-3(1+w_X)} + \Lambda - \frac{3K}{a^2}. \tag{5}$$

Note that the cosmological constant acts in the Friedmann equation like a new constituent whose density is constant with $a$, i.e., with $w_\Lambda = -1$. The spatial curvature acts like a constituent with $w_K = -\frac{1}{3}$. However, because the spatial curvature also appears directly in the metric, and not just through $a(t)$, it is distinguishable from a new contribution to the energy density. In contrast, $\Lambda$ really looks like a new material with $w = -1$.

It is common to re-write Eq. (5) by solving for $H$:

$$H = H_0 \sqrt{\sum_X \frac{\rho_{X0}}{\rho_{\rm c}} a^{-3(1+w_X)} + \frac{\Lambda}{3H_0^2} - \frac{K}{H_0^2 a^2}}, \tag{6}$$

where we have defined the *critical density* to be

$$\rho_{\rm c} = \frac{8\pi}{3H_0^2}. \tag{7}$$

This is the density the Universe would have to have today in the absence of spatial curvature or a cosmological constant. We define the density parameters

$$\Omega_X = \frac{\rho_X}{\rho_{\rm c}}, \quad \Omega_\Lambda = \frac{\Lambda}{3H_0^2}, \quad \text{and} \quad \Omega_K = -\frac{K}{H_0^2}. \tag{8}$$

Then we have

$$H = H_0 \sqrt{\sum_X \Omega_X a^{-3(1+w_X)} + \Omega_\Lambda + \Omega_K a^{-2}}. \tag{9}$$

By construction, the sum of the $\Omega$s – including $\Lambda$ and $K$ – is unity:

$$\sum_X \Omega_X + \Omega_\Lambda + \Omega_K = 1. \tag{10}$$

A common situation would be that we consider having matter and radiation, so that

$$H = H_0 \sqrt{\frac{\Omega_\mathrm{r}}{a^4} + \frac{\Omega_\mathrm{m}}{a^3} + \frac{\Omega_K}{a^2} + \Omega_\Lambda}. \tag{11}$$

Our observed Universe, as measured with the cosmic microwave background and galaxy surveys, appears to be consistent with $\Omega_K = 0$ (from *Planck*: $\Omega_K = -0.011^{+0.013}_{-0.012}$). The *Planck* satellite determination of those parameters [1] assuming a spatially flat Universe is $\Omega_\Lambda = 0.685 \pm 0.007$; $\Omega_\mathrm{m} = 0.315 \pm 0.007$; $\Omega_\mathrm{r}/\Omega_\mathrm{m} = (2.94 \pm 0.02) \times 10^{-4}$; and $H_0 = 67.4 \pm 0.5$ km/s/Mpc. Note, however, that direct measurements of $H_0$ prefer a larger value.

## III. EXPANSION HISTORY

We may now look at a few different limits of the expansion history of the Universe using Eq. (11).

First, we note that if all of $\Omega_{\mathrm{r,m},K,\Lambda}$ are $\geq 0$, then $H$ is real for all values of $a$: such a universe can expand continuously from $a \approx 0$ to $a \to \infty$. On the other hand, if one allows for negative $\Omega$s – for example, a closed Universe ($\Omega_K < 0$) with no cosmological constant ($\Lambda = 0$), or with negative cosmological constant ($\Lambda < 0$) – then there is a maximum possible value of $a$, $a_\mathrm{max}$. When the Universe expands to $a_\mathrm{max}$, we have $H = 0$ or $\dot{a} = 0$. In such a case, the Universe turns around and contracts again, taking the negative branch of Eq. (11), leading ultimately to a *Big Crunch*.

In general, we may solve for the expansion history by writing:

$$t = \int \frac{\mathrm{d}a}{\dot{a}} = \int \frac{\mathrm{d}a}{aH} = \frac{1}{H_0} \int \left[ \frac{\Omega_\mathrm{r}}{a^4} + \frac{\Omega_\mathrm{m}}{a^3} + \frac{\Omega_K}{a^2} + \Omega_\Lambda \right]^{-1/2} \frac{\mathrm{d}a}{a}. \tag{12}$$

In different parts of cosmic history, the contributions from different $\Omega$s may change: when $a$ is small, the radiation dominates, followed by a transition to matter domination and ultimately $\Lambda$-domination.

Of particular interest is the behavior of $a$ vs. $t$ in the regimes where the different constituents dominate:

- When radiation dominates, $t \propto \int a \, \mathrm{d}a \propto a^2$, so $a \propto t^{1/2}$.
- When matter dominates, $t \propto \int a^{1/2} \, \mathrm{d}a \propto a^{3/2}$, so $a \propto t^{2/3}$.
- If the Universe is open ($\Omega_K > 0$), then when curvature dominates, $t \propto \int \mathrm{d}a \propto a$, so $a \propto t$.
- When $\Lambda$ dominates, $H = \sqrt{\Lambda/3}$ so $a \propto \exp(\sqrt{\frac{\Lambda}{3}}\, t)$.

We can see that for the radiation-dominated case – which is relevant in the early Universe – $a$ reaches zero at a finite time $t$ (here taken to be $t = 0$). This is the *Big Bang*.

In the case of black holes, we found it useful to study the causal structure – that is, how information can propagate, and whether signals (light or matter) can go from one event to another. We can do this in cosmology by defining a new time coordinate, the *conformal time* $\eta$, where:

$$\eta = \int \frac{\mathrm{d}t}{a}. \tag{13}$$

Then the FLRW metric becomes

$$\mathrm{d}s^2 = [a(\eta)]^2 \left\{ -\mathrm{d}\eta^2 + \mathrm{d}\chi^2 + [r(\chi)]^2 (\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2) \right\}. \tag{14}$$

In this metric, if we choose a comoving observer $\mathcal{O}$ located at the origin (without loss of generality, us), then there are radially outgoing and ingoing rays that travel along lines of $\eta \pm \chi =$constant. One big difference between the black hole case and cosmology, though, is that in cosmology $\mathcal{O}$ is not special: because the universe is homogeneous and isotropic, any other comoving observer sees the same thing. But if our interest is in causal structure, we might as well choose one observer, and ask what they can see.

An interesting feature of the universe with radiation, matter, and $\Lambda$ is that the conformal time integral is convergent both at the Big Bang (because the integrand is proportional to $t^{-1/2}$, so $\int t^{-1/2}\,\mathrm{d}t$ converges at $t \to 0$) and at late times (because the integrand is proportional to $\exp(-\sqrt{\frac{\Lambda}{3}}\,t)$, so $\int \exp(-\sqrt{\frac{\Lambda}{3}}\,t)\,\mathrm{d}t$ converges at $t \to \infty$). This means that any observer can only see regions out to some maximum comoving distance (given by $\eta - \eta_{\rm BB}$, where $\eta_{\rm BB}$ is the conformal time at the Big Bang, $t = 0$) and can only transmit to regions out to some maximum comoving distance (given by $\eta_\infty - \eta$, where $\eta_\infty$ is the conformal time at $t = \infty$). Both of these two facts lead to some weird behaviors in cosmology. I will focus the remainder of this lecture on the issue of what happens in the future.

## IV. THE DISTANT FUTURE

In our Universe, which appears to have a positive $\Lambda$, the cosmological constant will dominate over matter and radiation and (if present) spatial curvature in the distant future. The scale factor really will increase exponentially; without loss of generality we set

$$a = \mathrm{e}^{H_\infty t}, \quad H_\infty = \sqrt{\frac{\Lambda}{3}}. \tag{15}$$

Then the metric is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{e}^{2H_\infty t}\left[\mathrm{d}\chi^2 + \chi^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right]. \tag{16}$$

This spacetime is most easily understood using the coordinate transformation

$$t = \tilde{t} + \frac{1}{2H_\infty}\ln(1 - H_\infty^2\tilde{r}^2), \quad \chi = \mathrm{e}^{-H_\infty t}\tilde{r}. \tag{17}$$

Then a comoving observer at position $\chi$ travels along a path of exponentially increasing $\tilde{r}$ (in fact, $\tilde{r} = a\chi$ is the distance from the origin to the observer measured along a slice of constant $t$). This transformation leads to the *de Sitter metric*

$$\mathrm{d}s^2 = -(1 - H_\infty^2\tilde{r}^2)\,\mathrm{d}\tilde{t}^2 + \frac{\mathrm{d}\tilde{r}^2}{1 - H_\infty^2\tilde{r}^2} + \tilde{r}^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{18}$$

This spacetime looks like an inside-out black hole: it has a horizon at $\tilde{r} = H_\infty^{-1}$, and the region $\tilde{r} < H_\infty^{-1}$ is visible to the observer $\mathcal{O}$. Every observer $\mathcal{O}$ thinks they are at the maximum of the gravitational potential, which is $-\frac{1}{2}H_\infty^2\tilde{r}^2$ at small $\tilde{r}$. Nothing special happens to a galaxy $\mathcal{G}$ that crosses the horizon, except that that is the last instant that $\mathcal{G}$ can send a signal to $\mathcal{O}$. In the distant future, classical GR states that $\mathcal{O}$ will see an exponentially redshifting image of $\mathcal{G}$, frozen in time at the instant $\mathcal{G}$ crossed the horizon. The area is $A = 4\pi\tilde{r}^2 = 12\pi\Lambda^{-1}$. This suggests that the de Sitter universe has a limiting entropy of

$$S_{\rm deS} = \frac{1}{4}A = \frac{3\pi}{\Lambda} \approx 3 \times 10^{122}. \tag{19}$$

The exact same argument that we gave for the law of increase of horizon area for black holes also applies in cosmology (we have to define $\mathcal{B}$ to be the region that can't send a signal to $\mathcal{O}$). What that means is that – if $\Lambda$ is really the correct explanation for cosmic acceleration – then in the future, all the other galaxies will accelerate away from the Local Group of galaxies (recall gravitationally bound objects such as the Local Group don't expand with the Universe). Those galaxies fall outside our horizon $\mathbb{H}$, leaving the Local Group all alone. Any piece of matter that is ejected from the Local Group and is swept away by the $-\frac{1}{2}H_\infty^2\tilde{r}^2$ potential falls through the horizon $\mathbb{H}$ and increases its area (hence entropy). In the very distant future, even the Local Group itself will disassemble, since individual particles can quantum mechanically tunnel out of the Local Group's gravitational potential and be swept away.

In a universe with a cosmological constant and with matter obeying the null energy condition, $S_{\rm deS}$ is the ultimate limiting value of entropy. This is our present-day conception of the heat death of the Universe.

---

[1] `https://arxiv.org/pdf/1807.06209.pdf`