Lecture XXI: Hawking radiation

(Dated: November 13, 2019)

I. OVERVIEW

In our previous lectures, we found that black holes have an area that can only grow with time, and moreover we had found a relation that looked suspiciously like the first law of thermodynamics with horizon area playing the role of entropy. We want to complete this study by showing that in fact black holes do radiate as blackbodies at some temperature T. This temperature is exactly the right quantity to fit in a thermodynamic relation, and it will tell us the constant of proportionality between area A and entropy S.

The treatment of this requires quantum field theory in curved spacetime; we will first do the classical field theory, and then paste quantum mechanics on as appropriate. For much more on this subject, see the monograph by Birrell & Davies, *Quantum fields in curved spacetime*.

II. SCALAR WAVES IN THE SCHWARZSCHILD SPACETIME

Let's begin this study by going back to the Schwarzschild spacetime,

$$\mathrm{d}s^2 = -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-2M/r} + r^2(\mathrm{d}\theta^2 + \sin^2\mathrm{d}\phi^2). \tag{1}$$

We will consider the real scalar wave equation in this spacetime, given by

$$\Box\psi \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta\psi = 0. \tag{2}$$

(I could do something more sophisticated such as electromagnetic or gravitational waves, or spinors, but this introduces indices that are not essential to the problem at hand. Scalars are simple and there is only one way to write their second derivative.)

The scalar wave equation simplifies to

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\psi - g^{\alpha\beta}\Gamma^\gamma{}_{\beta\alpha}\partial_\gamma\psi = 0, \tag{3}$$

or – plugging in the Christoffel symbols from Schwarzschild –

$$-\frac{1}{1-2M/r}\partial_t^2\psi + \left(1-\frac{2M}{r}\right)\partial_r^2\psi + \frac{1}{r^2}\partial_\theta^2\psi + \frac{1}{r^2\sin^2\theta}\partial_\phi^2\psi + \left(\frac{2}{r}-\frac{2M}{r^2}\right)\partial_r\psi + \frac{\cos\theta}{r^2\sin\theta}\partial_\theta\psi = 0. \tag{4}$$

This equation can undergo separation of variables. Let's write its multipole moments:

$$\psi(t,r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{r}\Psi_{\ell m}(t,r)Y_{\ell m}(\theta,\phi), \tag{5}$$

where we put in the factor of r in order to make the math simpler later (it is like putting in the factor of r in a 3D spherical potential quantum mechanics problem: it takes out the $\sim 1/r$ variation due power dilution in a spherical wave). The angular derivatives in Eq. (4) are $1/r^2$ times the angular component of the Laplacian, and thus give a factor of $-\ell(\ell+1)/r^2$:

$$-\frac{1}{1-2M/r}\partial_t^2\frac{\Psi_{\ell m}}{r} + \left(1-\frac{2M}{r}\right)\partial_r^2\frac{\Psi_{\ell m}}{r} + \left(\frac{2}{r}-\frac{2M}{r^2}\right)\partial_r\frac{\Psi_{\ell m}}{r} - \frac{\ell(\ell+1)}{r^2}\frac{\Psi_{\ell m}}{r} = 0. \tag{6}$$

The next step is to do the substitution from r to $\bar{r}$. We use:

$$\partial_r = \frac{\mathrm{d}\bar{r}}{\mathrm{d}r}\partial_{\bar{r}} = \frac{1}{1-2M/r}\partial_{\bar{r}}, \tag{7}$$

so that Eq. (6) multiplied by r becomes

$$-\frac{1}{1-2M/r}\partial_t^2\Psi_{\ell m} + r\partial_{\bar{r}}\left(\frac{1}{1-2M/r}\partial_{\bar{r}}\frac{\Psi_{\ell m}}{r}\right) + \left(2-\frac{2M}{r}\right)\frac{1}{1-2M/r}\partial_{\bar{r}}\frac{\Psi_{\ell m}}{r} - \frac{\ell(\ell+1)}{r^2}\Psi_{\ell m} = 0. \tag{8}$$

Using the product rule extensively and the fact that

$$\partial_{\bar r}\frac{1}{1-2M/r} = \left(1-\frac{2M}{r}\right)\partial_r\frac{1}{1-2M/r} = \frac{-2M/r^2}{1-2M/r} \quad \text{and} \quad \partial_{\bar r}\frac{1}{r} = -\frac{1}{r^2}\partial_{\bar r} r = -\frac{1-2M/r}{r^2}, \tag{9}$$

we get a long chain of simplifications. This results in

$$-\frac{1}{1-2M/r}\partial_t^2\Psi_{\ell m} + \frac{1}{1-2M/r}\partial_{\bar r}^2\Psi_{\ell m} - \left(\frac{2M}{r^3}+\frac{\ell(\ell+1)}{r^2}\right)\Psi_{\ell m} = 0. \tag{10}$$

We may multiply through by $1-2M/r$ and obtain:

$$-\partial_t^2\Psi_{\ell m} + \partial_{\bar r}^2\Psi_{\ell m} - \left(1-\frac{2M}{r}\right)\left(\frac{2M}{r^3}+\frac{\ell(\ell+1)}{r^2}\right)\Psi_{\ell m} = -\partial_t^2\Psi_{\ell m} + \partial_{\bar r}^2\Psi_{\ell m} - U_\ell(\bar r)\Psi_{\ell m} = 0, \tag{11}$$

which looks like a standard 1D wave equation with a potential. The potential goes exponentially to zero as $\bar r \to -\infty$ (recall: $1-2M/r \propto \mathrm{e}^{\bar r/2M}$) and to zero as $\sim 1/\bar r^2$ (or faster for $\ell = 0$) at large $\bar r$. Indeed, if $M \to 0$, this looks like a standard angular momentum barrier. Both far from the hole and near the horizon, this is a freely propagating wave equation! The problem is described by a power transmission coefficient, $\mathbb{T}_\ell(\omega)$ at frequency ω (i.e., for a wave at $\Psi \propto \mathrm{e}^{-\mathrm{i}\omega t}$), which is the fraction of the power incident from ∞ in the ℓm wave mode that is absorbed by the hole. In quantum mechanics, it is the probability that an incident particle in this partial wave is absorbed by the hole.

III. THE MIRROR FALLING INTO THE HOLE

Now let's imagine that a spherical mirror falls into the black hole in a spherically symmetric way. (This is idealized, but is intended to be the simplest way to get Hawking radiation. The key point is that all outgoing waves in the system arose from ingoing waves that just barely managed to make it back out before they crossed the horizon; the details don't affect that basic outcome.) The mirror is a boundary condition $\psi = 0$ or $\Psi_{\ell m} = 0$ at its world line in $(t, \bar r)$-space.

Once the mirror is far past the potential, the solution for $\Psi_{\ell m}$ is a simple ingoing and outgoing plane wave,

$$\Psi_{\ell m}(t, \bar r) = A_{\ell m\uparrow}(\bar\eta) + A_{\ell m\downarrow}(\bar\xi), \tag{12}$$

where $\bar\eta = t - \bar r$ and $\bar\xi = t + \bar r$, as usual. These describe the upward- and downward-going wave solutions near the hole. We must have

$$A_{\ell m\uparrow}(\bar\eta) = -A_{\ell m\uparrow}(\bar\xi_{\rm M}(\bar\eta)), \tag{13}$$

where $\bar\xi_{\rm M}(\bar\eta)$ is the mirror's world-line. Now in the (ξ, η)-diagram, the mirror crosses the horizon ($\eta = 0$) at some critical $\xi_{\rm c}$ with some slope $1/(q\xi_{\rm c})$. Therefore the world line satisfies

$$\bar\xi_{\rm M}(\bar\eta) = 4M\ln\xi = 4M\ln(\xi_{\rm c} + q\xi_{\rm c}\eta) = 4M\ln(\xi_{\rm c}(1 - q\mathrm{e}^{-\bar\eta/4M})) \approx K - 4Mq\mathrm{e}^{-\bar\eta/4M}, \tag{14}$$

where K is a constant. Therefore the outward-reflected signal is the inversion of the ingoing signal, but with the time axis remapped acoording to Eq. (14). Ingoing signals at $\bar\xi \geq K$ never reflect.

But of course, classically if the incident amplitude is zero, then nothing comes back. Quantum mechanically, however, we know that the scalar field has quantum fluctuations that go in. The big question for us, then, is what comes out.

IV. HAWKING RADIATION

We will now derive what happens to quantum fluctuations going into the hole. We know there is $\frac{1}{2}\hbar\omega$ of energy even in the vacuum state of every wave mode. This is true for even extraordinarily high frequencies. It will turn out that these fluctuations get turned into real outgoing modes. They are extraordinarily redshifted, and their redshift grows with time, but even when the redshift gets to some extraordinarily large value, there was always an extraordinarily high-frequency ingoing wave whose vacuum fluctuations are redshifted to the frequency of interest. These redshifted and virtual-turned-real waves are known as *Hawking radiation.*

A. Quantum fluctuations

Let's now consider quantum fluctuations. For a scalar field, the amplitudes falling into the black hole satisfy

$$\mathrm{Var}\, A_{\updownarrow} = \int \left[N_{\updownarrow}(\omega) + \frac{1}{2}\right] \hbar\omega \times \frac{\mathrm{d}\omega}{2\pi\omega^2}, \tag{15}$$

where the first part of the integrand is the energy per mode for a particle occupation number $N_{\updownarrow}$ at frequency ω. The second part has normalization factors: we divide by 2 since the energy is half kinetic and half potential; we divide by $\frac{1}{2}\omega^2$ since the kinetic energy density is $\frac{1}{2}(\partial_t\Psi)^2$, not $\frac{1}{2}\Psi^2$; and we divide by 2π since we have constructed an energy density per mode, and the number of ingoing modes per unit length (per unit $\bar{r}$) is $\Delta\omega/2\pi$. Now of course this is a variance; if we want the covariance at two different ingoing times, $\bar{\xi}$ and $\bar{\xi}'$, we must insert the appropriate phase factor:

$$\langle A_{\downarrow}(\bar{\xi})A_{\downarrow}(\bar{\xi}')\rangle = \int \left[N_{\downarrow}(\omega) + \frac{1}{2}\right] \hbar\omega \cos\omega(\bar{\xi}-\bar{\xi}')\,\frac{\mathrm{d}\omega}{2\pi\omega^2}. \tag{16}$$

(Since $A_{\downarrow}$ is real, we get only a cosine and not a complex exponential.) If vacuum fluctuations only are going down, so that $N_{\downarrow}(\omega)=0$, we have

$$\langle A_{\downarrow}(\bar{\xi})A_{\downarrow}(\bar{\xi}')\rangle = \frac{\hbar}{4\pi}\int_{\omega_{\mathrm{min}}}^{\infty} \cos\omega(\bar{\xi}-\bar{\xi}')\,\frac{\mathrm{d}\omega}{\omega}. \tag{17}$$

I inserted a range of frequencies here, but explicitly put in ω_{min} since the integral will diverge as $\omega_{\mathrm{min}} \to 0$. This is an *infrared divergence*, and if we are doing things right it shouldn't be a problem. We can understand the divergence by substituting $z=\omega(\bar{\xi}-\bar{\xi}')$, and then noting that the intergral approaches $\int \mathrm{d}z/z$ as $z\to 0$; thus there is a logarithmic divergence of the lower limit:

$$\int_{z_{\mathrm{min}}}^{\infty} \cos z\,\frac{\mathrm{d}z}{z} \to \ln C - \ln z_{\mathrm{min}} = -\ln\frac{z_{\mathrm{min}}}{C} \tag{18}$$

for some constant C, and so

$$\langle A_{\downarrow}(\bar{\xi})A_{\downarrow}(\bar{\xi}')\rangle = -\frac{\hbar}{4\pi}\ln\frac{\omega_{\mathrm{min}}|\bar{\xi}-\bar{\xi}'|}{C}. \tag{19}$$

B. The expectation value of outgoing waves

Now let's consider the outgoing waves. We have

$$\begin{aligned}\langle A_{\uparrow}(\bar{\eta})A_{\uparrow}(\bar{\eta}')\rangle &= \langle A_{\downarrow}(\bar{\xi}_{\mathrm{M}}(\bar{\eta}))A_{\downarrow}(\bar{\xi}_{\mathrm{M}}(\bar{\eta}'))\rangle \\ &= -\frac{\hbar}{4\pi}\ln\frac{4Mq\omega_{\mathrm{min}}|\mathrm{e}^{-\bar{\eta}/4M}-\mathrm{e}^{-\bar{\eta}'/4M}|}{C} \\ &= -\frac{\hbar}{4\pi}\left[-\frac{\bar{\eta}+\bar{\eta}'}{8M} + \ln\sinh\frac{|\bar{\eta}-\bar{\eta}'|}{8M} + \ln\frac{4M\omega_{\mathrm{min}}}{C}\right].\end{aligned} \tag{20}$$

Let's now subtract out the vacuum fluctuations, which are Eq. (17) but with $\bar{\xi},\bar{\xi}'\to\bar{\eta},\bar{\eta}'$:

$$\begin{aligned}\langle A_{\uparrow}(\bar{\eta})A_{\uparrow}(\bar{\eta}')\rangle_{\mathrm{vac.\,subtr.}} &= -\frac{\hbar}{4\pi}\left[-\frac{\bar{\eta}+\bar{\eta}'}{8M} + \ln\sinh\frac{|\bar{\eta}-\bar{\eta}'|}{8M} + \ln\frac{4M\omega_{\mathrm{min}}}{C} - \ln\frac{\omega_{\mathrm{min}}|\bar{\eta}-\bar{\eta}'|}{C}\right] \\ &= -\frac{\hbar}{4\pi}\left[-\frac{\bar{\eta}+\bar{\eta}'}{8M} + \ln\frac{\sinh[|\bar{\eta}-\bar{\eta}'|/8M]}{|\bar{\eta}-\bar{\eta}'|/8M} - \ln 2\right].\end{aligned} \tag{21}$$

Note that this result is now finite. In the expression in brackets, the last term is a constant, so only contributes at zero frequency, and the first term – being a linear function of $\bar{\eta}$ and $\bar{\eta}'$ – will again contribute at zero frequency. So we don't care about them if we are interested in outgoing radiation. But the second term does something special.

C. The spectrum of Hawking radiation

We want to equate $\langle A_\uparrow(\bar\eta)A_\uparrow(\bar\eta')\rangle_{\rm vac.\,subtr.}$ to an integral of the type Eq. (16) without the $+\frac{1}{2}$ contribution due to vacuum fluctuations (which we have already subtracted). It is easiest to do this if we use the identity:

$$\frac{\sinh x}{x} = \prod_{n=1}^{\infty}\left[1+\frac{x^2}{(\pi n)^2}\right], \tag{22}$$

which is true since $\sinh x/x = 1$ at $x=0$, it has no singularities, and it has zeroes at $\pm\pi {\rm i}n$ for $n=1,2,3,\ldots$. This means

$$-\frac{\hbar}{4\pi}\ln\frac{\sinh[|\bar\eta-\bar\eta'|/8M]}{|\bar\eta-\bar\eta'|/8M} = -\frac{\hbar}{4\pi}\sum_{n=1}^{\infty}\ln\left[1+\frac{\Delta\bar\eta^2}{64\pi M^2n^2}\right], \tag{23}$$

where we set $\Delta\bar\eta\equiv\bar\eta-\bar\eta'$.

On the other hand, if we define

$$f_\mu(\Delta\bar\eta) = \int_0^\infty {\rm e}^{-\mu\omega}\cos\omega\Delta\bar\eta\,\frac{{\rm d}\omega}{\omega}, \tag{24}$$

then

$$\begin{aligned}\partial_{\Delta\bar\eta}f_\mu(\Delta\bar\eta) &= -\int_0^\infty {\rm e}^{-\mu\omega}\sin\omega\Delta\bar\eta\,{\rm d}\omega\\ &= \Im\int_0^\infty {\rm e}^{-(\mu+{\rm i}\Delta\bar\eta)\omega}\,{\rm d}\omega\\ &= \Im\frac{1}{\mu+{\rm i}\Delta\bar\eta}\\ &= -\frac{\Delta\bar\eta}{\mu^2+\Delta\bar\eta^2}\\ &= -\frac{1}{2}\partial_{\Delta\bar\eta}\ln(\mu^2+\Delta\bar\eta^2)\\ &= -\frac{1}{2}\partial_{\Delta\bar\eta}\ln\left(1+\frac{\Delta\bar\eta^2}{\mu^2}\right).\end{aligned} \tag{25}$$

Integrating both sides, and recalling that the additive constant is another zero-frequency contribution, we see that each term in Eq. (23) is an f_μ with $\mu=8\pi Mn$:

$$\langle A_\uparrow(\bar\eta)A_\uparrow(\bar\eta')\rangle_{\rm vac.\,subtr.} = \frac{\hbar}{2\pi}\sum_{n=1}^{\infty}f_{8\pi Mn}(\Delta\bar\eta) + \text{[zero frequency terms]}. \tag{26}$$

Comparison to Eq. (16) shows that this is the spectrum of radiation whose occupation number is an infinite sum of exponentials, i.e., a geometric series:

$$N_\uparrow(\omega) = \sum_{n=1}^{\infty}{\rm e}^{-8\pi Mn\omega} = \frac{1}{{\rm e}^{8\pi M\omega}-1}. \tag{27}$$

This is a blackbody spectrum at temperature given by

$$T_{\rm H} = \frac{\hbar}{8\pi k_{\rm B}M} = \frac{\hbar c^3}{8\pi k_{\rm B}GM}. \tag{28}$$

Thus the horizon appears as a blackbody with temperature $T_{\rm H}$. A black hole is not perfectly black! Note that – due to the appearance of $\hbar$ – this is a purely quantum phenomenon.

The blackbody temperature of a black hole in astrophysical units is

$$T_{\rm H} = 60\left(\frac{1\,M_\odot}{M}\right)\,{\rm nK}, \tag{29}$$

which is small enough that we don't need to worry about it for astrophysical purposes, and we have no way of measuring it in any practical circumstance. However, this thought experiment is very important for fundamental physics and probably for quantum gravity.

In what follows, I will set $\hbar$ and $k_{\rm B}$ to 1, i.e., I will use Planck units.

V. THERMODYNAMICS, STATISTICAL MECHANICS, AND EVAPORATION

Let's return to the thermodynamic identity,

$$\mathrm{d}M = \mathrm{d}T\,\mathrm{d}S + \Omega\,\mathrm{d}J. \tag{30}$$

For the Schwarzschild black hole ($J = 0$), if we interpret T_H as the temperature T, we see that the entropy satisifies

$$S = \int \frac{\mathrm{d}M}{T} = \int 8\pi M\,\mathrm{d}M = 4\pi M^2 = \frac{1}{4}A, \tag{31}$$

where A is the horizon area. So there is indeed an entropy proportional to the horizon area, with a factor of $\frac{1}{4}$. This is consistent with the interpretation of the area theorem in terms of thermodynamics.

This suggests that there is some microphysical degree of freedom that stores one "bit" ($\Delta S = \ln 2$) on the horizon for every $4\ln 2$ square Planck lengths of horizon area (i.e., for every $7.3\times 10^{-70}\,\mathrm{m}^2$). The true nature of these degrees of freedom remains unknown. But again, this is probably a big hint in terms of how to construct a consistent theory of quantum gravity.

Another remarkable implication of Hawking radiation is that a black hole will evaporate: since it is a blackbody, it will lose mass at a rate of

$$-\dot{M} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_0^\infty \frac{1}{\mathrm{e}^{8\pi M\omega}-1}\,\omega\mathbb{T}_\ell\,\frac{\mathrm{d}\omega}{2\pi}, \tag{32}$$

where we have summed over modes, introduced the barrier transmission probability $\mathbb{T}_\ell$ (only particles that pass through the barrier will escape), and put in the energy per particle ω and flux of modes $\mathrm{d}\omega/2\pi$. This is finite because the barrier gets higher with ℓ and so at large ℓ, $\mathbb{T}_\ell \to 0$. Also this is for a scalar particle; for other types of particles (e.g., photons or gravitons) one also needs a polarization sum and the correct transmission probability. For small enough (hot enough) black holes, one further needs to include massive particles such as neutrinos or even electrons/positrons.

The result from Eq. (32) can be estimated at order of magnitude to be $\sim 1/M^2$ (the sums give factors of order unity and the integrand peaks when $\omega \sim 1/M$). Therefore a black hole is expected to have a lifetime of order $\sim M^3$ (in Planck units). A stellar-mass black hole has a mass of $\sim 10^{38}$ Planck masses, so a lifetime of $\sim 10^{114}$ Planck times or 10^{64} years (assuming it doesn't eat anything; in the present-day Universe, it is eating CMB photons even if there is nothing else available, so it is still growing). But these black holes are not forever. It has been estimated that black holes born in the early Universe at $M < 5\times 10^{11}$ kg would have evaporated by today [see MacGibbon, PRD 44:376 (1991)].

Quantum mechanics is unitary, and if taken literally this would suggest that the radiation emitted by a black hole is not just random thermal radiation: it should encode the state of the matter that went into making the black hole. It is generally expected that this is true in a full theory of quantum gravity, but of course we don't know for sure!