Lecture XIX: Angular momentum and rotating black holes

(Dated: November 6, 2019)

I. OVERVIEW

We have discussed the Schwarzschild solution, which describes a spherically symmetric black hole. But many astrophysical black holes have angular momentum. Angular momentum, like energy, is a conserved quantity – if a black hole swallows some angular momentum, that is part of the black hole, and the spacetime surrounding it must show some manifestation of the presence of that angular momentum. We will discuss the solutions with angular momentum in this lecture. We will begin with a discussion of angular momentum in linearized gravity, and then discuss the Kerr black hole (the solution for a black hole with mass and angular momentum).

Before we look at black holes, let's briefly consider how angular momentum works in linearized gravity. Recall the linearized gravity equation:

$$\Box \bar{h}^{\mu\nu} = -16\pi T^{\mu\nu}. \tag{1}$$

Let's now look at the $0i$ part of this equation:

$$\Box h^{0i} = \Box \bar{h}^{0i} = -16\pi T^{0i}, \tag{2}$$

where we used that h^{0i} is off-diagonal so there is no difference between h^{0i} and $\bar{h}^{0i}$.

A. Metric description

Now let's specialize to the case of a stationary problem, or at least to considering the DC part of the metric perturbation (the AC part we discussed when we looked at gravitational waves). We will also assume a compact source, so that far away we can do a power series expansion in the size of the source. In this case, the $\Box$ becomes a ∇^2, and the solution for h^{0i} is the solution to the Poisson equation (just like an electrostatics problem):

$$h^{0i}(\boldsymbol{r}) = 4\int \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\, T^{0i}(\boldsymbol{r}')\,\mathrm{d}^3\boldsymbol{r}'. \tag{3}$$

This equation generates h^{0i} from T^{0i} (momentum density) in much the same way that in magnetostatics the vector potential A^i is generated from the current density J^i. Therefore this phenomenon is often called *gravitomagnetism*. If we expand this in powers of $\boldsymbol{r}'$, working to first order, we get:

$$h^{0i}(\boldsymbol{r}) = 4\int \left(\frac{1}{r} + \frac{\hat{r}^j r'^j}{r^2} + ...\right) T^{0i}(\boldsymbol{r}')\,\mathrm{d}^3\boldsymbol{r}' \approx 4\frac{1}{r}\int T^{0i}(\boldsymbol{r}')\,\mathrm{d}^3\boldsymbol{r}' - \frac{4}{r^2}\hat{r}^j\int r'^j T^{0i}(\boldsymbol{r}')\,\mathrm{d}^3\boldsymbol{r}'. \tag{4}$$

The first integral is the total linear momentum; for the purposes of this problem, I will work in the center of mass frame so that is zero. For the second integral, we define the moment

$$M^{ij} \equiv \int r'^j T^{0i}(\boldsymbol{r}')\,\mathrm{d}^3\boldsymbol{r}', \tag{5}$$

so

$$h^{0i}(\boldsymbol{r}) = \frac{4}{r^2}\hat{r}^j M^{ij}. \tag{6}$$

This is of course the leading-order piece; there is also an order $1/r^3$ term, depending on one higher moment of the momentum density, etc.

Now the moments M^{ij} have units of position times momentum. The antisymmetric piece is the angular momentum:

$$J_k = \int \varepsilon_{kji} r'^j T^{0i}(\boldsymbol{r}')\,\mathrm{d}^3\boldsymbol{r}' = \varepsilon_{kji}M^{ij} \quad \leftrightarrow \quad M^{[ij]} = \frac{1}{2}\varepsilon^{kji}J_k. \tag{7}$$

It looks from Eq. (6) as if all of M^{ij} (both symmetric and antisymmetric pieces) contribute to the $0i$ metric perturbation, but in fact the symmetric part can be removed by a gauge transformation. Recall that in an infinitesimal gauge transformation in linearized gravity, generated by changing the coordinates by α, we have

$$\Delta h^{\alpha\beta} = \xi^{\alpha,\beta} + \xi^{\beta,\alpha}. \tag{8}$$

So if we have a perturbation with $\xi^0 = \psi(\boldsymbol{r})$ (only spatial dependence), and $\xi^i = 0$, then only the $0i$ and $i0$ components of $h^{\alpha\beta}$ will change, and they will change by

$$\Delta h^{0i} = \partial_i \psi(\boldsymbol{r}). \tag{9}$$

Inspection shows that by choosing a function of the form $\psi = C_{kl}\hat{r}^k\hat{r}^l/r$, with $\mathbf{C}$ a symmetric 3×3 matrix, it is possible to cancel out the terms in Eq. (6) coming from the symmetric part of $\mathbf{M}$. Thus only the antisymmetric part survives, and we write

$$h^{0i}(\boldsymbol{r}) = \frac{4}{r^2}\hat{r}^j M^{[ij]} = -\frac{2}{r^2}\hat{r}^j\varepsilon^{kji}J_k. \tag{10}$$

This contribution cannot be removed by a gauge transformation; the above trick with ψ doesn't work since there is a "curl" component to h^{0i}.

Again in weak gravity situations, I could take this perturbation, lower the indices (which turns the $-$ into a $+$ sign), and see that it contributes a change to $\mathrm{d}s^2$ of:

$$\mathrm{d}s^2 + = -\frac{4}{r^2}\hat{r}^j\varepsilon^{kji}J_k\,\mathrm{d}t\,\mathrm{d}x^i = -\frac{4}{r}J\sin^2\theta\,\mathrm{d}t\,\mathrm{d}\phi, \tag{11}$$

where in the last equality I put $\mathbf{J}$ on the z-axis and wrote the expression in spherical coordinates. (You can show this explicitly by considering $\phi = 0$, so $\hat{r}^1 = \sin\theta$ and $\mathrm{d}x^2 = r\sin\theta\,\mathrm{d}\phi$.) In the case of a rotating star with angular momentum $\mathbf{J}$, then, we expect that far from the star the metric will be

$$\mathrm{d}s^2 \approx -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-2M/r} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) - \frac{4}{r}J\sin^2\theta\,\mathrm{d}t\,\mathrm{d}\phi. \tag{12}$$

This is only true in the limit of being far from the star; in general, a rotating star may have higher-order multipole moments (e.g., a quadrupole moment). Moreover, to have a vacuum solution, we will need to include higher orders in J. But the term involving J – which is a correction of order $1/r^2$ (it looks like a $1/r$ in polar coordinates, but you should think of it as $-4J\sin\theta/r^2 \times \mathrm{d}t \times r\sin\theta\,\mathrm{d}\phi$) – is there, and is an important gravitational effect from moving matter. The angular momentum is the gravitational analogue of the magnetic moment in electrodynamics.

In GR, the $\mathrm{d}t\,\mathrm{d}\phi$ term far from an object is normally taken as the definition of the total angular momentum, in much the same way that Kepler's 3rd law is the definition of total mass. The effect of this term on orbiting objects will be considered soon. (Its effect in the case of the rotating Earth was measured by the Gravity Probe B satellite, which measured the effect of J on the precession of an orbiting gyroscope.)

II. THE KERR BLACK HOLE

We may now search for solutions for black holes that have angular momentum. These should be vacuum spacetimes ($T^{\mu\nu} = 0$), which approach Eq. (12) far away from the hole. In general, a black hole that forms and is not exactly spherical will have perturbations of all multipole moments, but the quadrupole and higher moments lead to propagating gravitational waves; it can be shown that the result is a black hole completely described by its mass M and angular momentum J. (This was first shown in the 1970s.) There is one resulting solution of this type: the *Kerr metric*, which turns out to be:

$$\mathrm{d}s^2 = -\frac{\Delta - a^2\sin^2\theta}{\rho^2}\mathrm{d}t^2 - 2\frac{2aMr\sin^2\theta}{\rho^2}\mathrm{d}t\,\mathrm{d}\phi + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta\,\mathrm{d}\phi^2 + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\,\mathrm{d}\theta^2, \tag{13}$$

where $a = J/M$ and we have defined the auxiliary quantities

$$\Delta = r^2 - 2Mr + a^2 \quad \text{and} \quad \rho^2 = r^2 + a^2\cos^2\theta. \tag{14}$$

(This is actually the Boyer-Lindquist coordinate system, which is the most convenient for computing geodesics but not the original form of the Kerr metric.) Inspection shows that this becomes the Schwarzschild metric if $J = 0$, and that the $\mathrm{d}t\,\mathrm{d}\phi$ term is indeed that of Eq. (12). It is a remarkable fact that there is an analytic expression for it at all.

A. Coordinate singularities

In the Kerr metric, $\rho^2 > 0$ for all $r > 0$. There is a trivial coordinate singularity at $\theta = 0, \pi$. One can see that there is a singularity of the form $g_{rr} \to \infty$ at $\Delta = 0$, or at

$$r = r_\pm = M \pm \sqrt{M^2 - a^2}. \tag{15}$$

Less obvious is that the $t\phi$ sector of the metric (which is not diagonal) also has a singularity at $\Delta = 0$. This is because even though the metric components g_{tt}, $g_{\phi\phi}$, and $g_{t\phi}$ remain finite, the determinant of this 2×2 piece has – after a lot of algebraic simplification –

$$g_{tt}g_{\phi\phi} - g_{t\phi}^2 = -\Delta \sin^2\theta. \tag{16}$$

The coordinate singularities that occur at $r = r_\pm$ are just like the coordinate singularity in the Schwarzschild metric at $r = 2M$, aside from the fact that the zero eigenvector of $g_{\mu\nu}$ is no longer in the t-direction but is some linear combination of t and ϕ. This means that those coordinate singularities can be dealt with in basically the same way; they correspond to event horizons. The outer event horizon (r_+) is the event horizon from an astronomer's point of view; from a general relativity point of view, both r_+ and r_- will turn out to be event horizons, but there are no observable (to us) implications of what happens near r_-.

In the non-rotating limit, $a \to 0$, we have $r_+ \to 2M$. But as the hole spins up, in the limit of $a_\star \equiv a/M \to 1$, we have $r_+ \to M$. For larger values of $a_\star$ the horizon disappears; as we will see later, several thought experiments suggest that a physical black hole in GR cannot be spun up to $a_\star \geq 1$. A black hole with $a_\star \to 1$ is thus called *maximally spinning.*

For future reference, we use the determinant relation of Eq. (16) to see that

$$g^{tt} = -\frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Delta\rho^2}, \quad g^{\phi\phi} = \frac{\Delta - a^2\sin^2\theta}{\Delta\rho^2\sin^2\theta}, \quad g^{t\phi} = -\frac{2aMr}{\Delta\rho^2}, \quad g^{rr} = \frac{\Delta}{\rho^2}, \quad \text{and} \quad g^{\theta\theta} = \frac{1}{\rho^2}. \tag{17}$$

B. Stationary and zero angular momentum observers

In the Schwarzschild metric, it was obvious how to think of a "stationary" observer. In the Kerr metric, one can define two useful types of observers that travel at constant r and θ. One is stationary with respect to the external universe, and has ϕ =constant. This observer has $u^\phi = 0$ and in general their 4-velocity is

$$u^\alpha \to \left((-g_{tt})^{-1/2}, 0, 0, 0\right) = \left(\sqrt{\frac{\rho^2}{\Delta - a^2\sin^2\theta}}, 0, 0, 0\right). \tag{18}$$

This observer we will refer to as "stationary." However, we note that a stationary observer is not possible everywhere outside the black hole (i.e., at $r > r_+$). The stationary observer requires $\Delta - a^2\sin^2\theta > 0$, i.e.,

$$r^2 - 2Mr + a^2\cos^2\theta > 0. \tag{19}$$

This is satisfied if

$$r > r_{\rm st}(\theta) = M + \sqrt{M^2 - a^2\cos^2\theta}. \tag{20}$$

(There is another solution inside the inner horizon that is not of interest to us.) Note that $r_{\rm st}(\theta) > r_+$ for $\theta \neq 0$. This surface is called the *static limit*, and inside of it there is no observer who can remain at constant longitude; all observers are dragged around, rotating in the direction of rotation of the hole! At the equator, the static limit reaches out to $2M$. The region between the outer event horizon and the static limit is known as the *ergosphere.* Because it is outside the event horizon, the ergosphere is observable to us and of great importance in black hole astrophysics.

To solve the problem of having reference observers in the ergosphere, we may define another type of observer, the *zero angular momentum observer* (ZAMO). The ZAMO has r and θ constant, but $u_\phi = 0$ (no z-angular momentum). Then

$$u_\alpha \to \left(-(-g^{tt})^{-1/2}, 0, 0, 0\right), \tag{21}$$

which is well-behaved at all $r > r_+$. The upper-index 4-velocity is

$$u^\alpha \to \left((-g^{tt})^{1/2}, \frac{-g^{t\phi}}{(-g^{tt})^{-1/2}}, 0, 0 \right). \tag{22}$$

Since $g^{tt} < 0$ for all cases outside the horizon ($r > r_+$), the ZAMO is always well-behaved. The ZAMO is rotating around the hole at angular velocity (as seen at ∞):

$$\Omega_{\rm ZAMO} = \frac{u^\phi}{u^t} = \frac{g^{t\phi}}{g^{tt}} = \frac{2aMr}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}. \tag{23}$$

C. The allowed trajectories for particles

One may ask at a given (r, θ) what is the range of allowed $\mathrm{d}\phi/\mathrm{d}t$. We can see that this is given by the restriction

$$g_{tt} + 2g_{t\phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} + g_{\phi\phi}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 < 0, \tag{24}$$

or $\Omega_{\rm min} < \frac{\mathrm{d}\phi}{\mathrm{d}t} < \Omega_{\rm max}$ with

$$\Omega_{\rm max,min} = \frac{-g_{t\phi}}{g_{\phi\phi}} \pm \frac{\sqrt{(g_{t\phi})^2 - g_{tt}g_{\phi\phi}}}{g_{\phi\phi}} = \Omega_{\rm ZAMO} \pm \frac{\Delta^{1/2}\sin\theta}{g_{\phi\phi}}. \tag{25}$$

What is interesting is that as we approach the outer event horizon, $g_{\phi\phi}$ remains finite but $\Delta \to 0$: thus all particles near the horizon must have

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} \approx \Omega_{\rm H} = \Omega_{\rm ZAMO}(r_+) = \frac{2aMr_+}{(r_+^2+a^2)^2} = \frac{a}{2M\left(M+\sqrt{M^2-a^2}\right)}. \tag{26}$$

This is 0 for a Schwarzschild black hole ($a = 0$), but approaches $\frac{1}{4M}$ for the maximally spinning case. We may call $\Omega_{\rm H}$ the angular velocity of the horizon.

An even stranger phenomenon occurs when we ask what is the allowed energy of particles going in the forward light cone. Writing $g^{\alpha\beta}p_\alpha p_\beta \le 0$, and taking $\mathcal{E} = -p_t$ and $\mathcal{L} = p_\phi$ (the energy and z-angular momentum), we find

$$g^{tt}\mathcal{E}^2 - 2g^{t\phi}\mathcal{E}\mathcal{L} + g^{\phi\phi}\mathcal{L}^2 \le -g^{rr}(u_r)^2 - g^{\theta\theta}(u_\theta)^2 \le 0. \tag{27}$$

Therefore – at a given angular momentum $\mathcal{L}$ – the energy is bounded by

$$\mathcal{E} \ge \mathcal{E}_{\rm min} = \frac{-g^{t\phi}}{-g^{tt}}\mathcal{L} + \frac{\sqrt{(g^{t\phi})^2 - g^{tt}g^{\phi\phi}}}{-g^{tt}}|\mathcal{L}|, \tag{28}$$

where we used $g^{tt} < 0$ outside the horizon. I took the positive root since that is the branch where the particle goes forward in time; the negative root would be the maximum energy for a particle going backward in time. Now since $g^{t\phi} < 0$ (for $J > 0$), we can see that for positive angular momentum $\mathcal{L} > 0$, $\mathcal{E}_{\rm min}$ is positive. But for negative angular momentum, we have

$$\mathcal{E}_{\rm min} = \frac{-|g^{t\phi}| + \sqrt{(g^{t\phi})^2 - g^{tt}g^{\phi\phi}}}{-g^{tt}}|\mathcal{L}|. \tag{29}$$

This minimum energy becomes negative when $g^{\phi\phi} < 0$, because the argument of the square root is larger than $(g^{t\phi})^2$. This occurs when $\Delta < a^2\sin^2\theta$, i.e., inside the ergosphere. So one can have particles on negative-energy trajectories! If such a particle falls into the black hole, the black hole will lose mass. Since this is possible only for negative angular momentum, the black hole will lose angular momentum as well. This process is called the *Penrose process*, and in principle is an energy source because the black hole's total energy (i.e., mass) is decreased in the process.

III. GEODESICS IN THE KERR SPACETIME

We now consider the possible geodesics in the Kerr spacetime.

A. Constants of the motion

It is obvious that the metric coefficients do not depend on t or ϕ. Therefore, p_t and p_ϕ are constant, and we may define the energy per unit mass $\tilde{\mathcal{E}} = -u_t = -p_t/\mu$ and the z-angular momentum per unit mass $\tilde{\mathcal{L}} = u_\phi = p_\phi/\mu$ (for mass μ), just as for the Schwarzschild case; also $\boldsymbol{u}\cdot\boldsymbol{u} = -1$ is a constant of the motion. Indeed this is true for any stationary axisymmetric spacetime. But since we don't have spherical symmetry, we can't just rotate the coordinates and put any orbit in the equatorial plane. The stationary axisymmetric spacetimes thus divide into two categories: there are spacetimes with an additional constant of the motion, in which case we can apply the usual separation of variables machinery and derive regular orbits; and spacetimes without an additional constant of the motion, in which case some orbits are chaotic (think of the Sun + Jupiter + asteroid problem).

It turns out that the Kerr metric does have an additional constant of the motion, which is totally not obvious: the *Carter constant*. To see how this works, recall from Lecture IX that for a geodesic

$$\dot{u}_\gamma = \frac{1}{2} g_{\mu\nu,\gamma} u^\mu u^\nu. \tag{30}$$

Now recall that from matrix inversion:

$$g^{\alpha\mu} g_{\mu\nu} = \delta^\alpha{}_\nu \quad \to \quad g^{\alpha\mu}{}_{,\gamma} g_{\mu\nu} + g^{\alpha\mu} g_{\mu\nu,\gamma} = 0. \tag{31}$$

This means that we can replace $g_{\mu\nu,\gamma}$ in Eq. (30):

$$\dot{u}_\gamma = \frac{1}{2} g_{\mu\nu,\gamma} u^\mu u^\nu = \frac{1}{2} g^{\alpha\mu} g_{\mu\nu,\gamma} u_\alpha u^\nu = -\frac{1}{2} g^{\alpha\mu}{}_{,\gamma} g_{\mu\nu} u_\alpha u^\nu = -\frac{1}{2} g^{\alpha\mu}{}_{,\gamma} u_\alpha u_\mu. \tag{32}$$

(So far this is general.) The case of interest to us in the Kerr metric will be the derivative of u_θ:

$$\dot{u}_\theta = -\frac{1}{2} g^{\alpha\mu}{}_{,\theta} u_\alpha u_\mu = -\frac{1}{2\rho^2}\left[(\rho^2 g^{\alpha\mu})_{,\theta} u_\alpha u_\mu - (\rho^2)_{,\theta} g^{\alpha\mu} u_\alpha u_\mu\right] = -\frac{1}{2\rho^2}\left[(\rho^2 g^{\alpha\mu})_{,\theta} u_\alpha u_\mu - 2a^2 \sin\theta\cos\theta\right], \tag{33}$$

where in the last step we used the normalization of $\boldsymbol{u}\cdot\boldsymbol{u} = -1$.

Now consider the quantity:

$$\tilde{\mathcal{Q}} = u_\theta^2 + \cos^2\theta \left[a^2(1-\tilde{\mathcal{E}}^2) + \frac{\tilde{\mathcal{L}}^2}{\sin^2\theta}\right]. \tag{34}$$

Its derivative is

$$\begin{aligned}
\frac{\mathrm{d}\tilde{\mathcal{Q}}}{\mathrm{d}\tau} &= 2u_\theta \dot{u}_\theta - 2\dot{\theta}\left[\sin\theta\cos\theta\, a^2(1-\tilde{\mathcal{E}}^2) + \frac{\cos\theta}{\sin^3\theta}\tilde{\mathcal{L}}^2\right] \\
&= -u_\theta \frac{1}{\rho^2}\left[(\rho^2 g^{\alpha\mu})_{,\theta} u_\alpha u_\mu - 2a^2\sin\theta\cos\theta\right] - 2\dot{\theta}\left[\sin\theta\cos\theta\, a^2(1-\tilde{\mathcal{E}}^2) + \frac{\cos\theta}{\sin^3\theta}\tilde{\mathcal{L}}^2\right] \\
&= -2\dot{\theta}\left[\frac{1}{2}(\rho^2 g^{\alpha\mu})_{,\theta} u_\alpha u_\mu - a^2\sin\theta\cos\theta + \sin\theta\cos\theta\, a^2(1-\tilde{\mathcal{E}}^2) + \frac{\cos\theta}{\sin^3\theta}\tilde{\mathcal{L}}^2\right] \\
&= -2\dot{\theta}\left[\frac{1}{2}(\rho^2 g^{\alpha\mu})_{,\theta} u_\alpha u_\mu - a^2\tilde{\mathcal{E}}^2\sin\theta\cos\theta + \frac{\cos\theta}{\sin^3\theta}\tilde{\mathcal{L}}^2\right] \\
&= -2\dot{\theta}\left[\frac{1}{2}(\rho^2 g^{tt})_{,\theta}\tilde{\mathcal{E}}^2 + \frac{1}{2}(\rho^2 g^{\phi\phi})_{,\theta}\tilde{\mathcal{L}}^2 - a^2\tilde{\mathcal{E}}^2\sin\theta\cos\theta + \frac{\cos\theta}{\sin^3\theta}\tilde{\mathcal{L}}^2\right] = 0,
\end{aligned} \tag{35}$$

where in the third line we used $u_\theta = \rho^2\dot{\theta}$; in the fourth line we used that $\rho^2 g^{\alpha\mu}$ depends on θ only for the tt and $\phi\phi$ components; and then we did the derivatives and found all terms cancel. Thus $\mathcal{Q}$ is an additional constant of the motion. For equatorial orbits, where $\cos\theta = 0$ and $\dot{\theta} = 0$, we see that $\mathcal{Q} = 0$.

The Carter constant allows one to solve for the turning points in θ where $\dot{\theta}$ (hence u_θ^2) goes to zero, which can be thought of as defining an "inclination" for the orbit. However, it is most interesting to us to consider the equatorial orbits. These are both the simplest, and illustrate some of the key features of the Kerr spacetime, and also are the most relevant to accretion discs. This is because inclined orbits in the Kerr spacetime precess (the period of oscillation in θ is not the same as in ϕ), and therefore a radiatively efficient disc of gas orbiting the black hole will undergo self-collisions and radiate away energy until it settles into a disc in the equatorial plane of the hole.

B. The equatorial orbits

We now consider the equatorial orbits with $\mathcal{Q} = 0$, and then $\theta = \pi/2$ and $u_\theta = 0$. These will be described by $\mathcal{L}$ and $\mathcal{E}$; the normalization $g^{\alpha\beta}u_\alpha u_\beta = -1$ then gives

$$-\frac{(r^2+a^2)^2 - a^2\Delta}{\Delta\rho^2}\tilde{\mathcal{E}}^2 + \frac{\Delta - a^2}{\Delta\rho^2}\tilde{\mathcal{L}}^2 + \frac{4aMr}{\Delta\rho^2}\tilde{\mathcal{E}}\tilde{\mathcal{L}} + \frac{\Delta}{\rho^2}(u_r)^2 = -1. \tag{36}$$

Multiplying by $\Delta\rho^2$, rearranging, and noting that $\rho^2 = r^2$ at the equator gives

$$\Delta^2(u_r)^2 = -\Delta r^2 + [(r^2+a^2)^2 - a^2\Delta]\tilde{\mathcal{E}}^2 - (\Delta - a^2)\tilde{\mathcal{L}}^2 - 4aMr\tilde{\mathcal{E}}\tilde{\mathcal{L}} \equiv P(r). \tag{37}$$

The right-hand side is a 4th order polynomial in r; there is an "allowed" region in r when $P(r) \geq 0$. At the outer horizon, we have $\Delta = 0$ and $r^2 + a^2 = 2Mr_+$, so

$$P(r_+) = 4M^2r_+^2\tilde{\mathcal{E}}^2 - 4aMr\tilde{\mathcal{E}}\tilde{\mathcal{L}} + a^2\tilde{\mathcal{L}}^2 = (2Mr_+\tilde{\mathcal{E}} - a\tilde{\mathcal{L}})^2 \geq 0, \tag{38}$$

so the outer horizon is always allowed. Also as $r \to \infty$, $P(r) \to (\mathcal{E}^2 - 1)r^4$, so $r \to \infty$ is allowed if and only if $\tilde{\mathcal{E}}^2 \geq 1$. Finally, one can see that $P(0) = 0$, so there is always a root at zero, and hence at most 3 roots between r_+ and ∞. This means that the classification of orbits into Cases A (3 roots in $r_+ < r < \infty$), B (1 root), C (2 roots), and D (0 roots) is similar to the Schwarzschild case.

C. Circular equatorial orbits

The boundaries between the cases are determined by the circular orbits, where $P(r) = P'(r) = 0$. Expanding $P(r)$ in powers of r gives:

$$P(r) = (\tilde{\mathcal{E}}^2 - 1)r^4 + 2Mr^3 + [a^2(\tilde{\mathcal{E}}^2 - 1) - \tilde{\mathcal{L}}^2]r^2 + 2M(\tilde{\mathcal{L}} - a\tilde{\mathcal{E}})^2 r. \tag{39}$$

Taking the derivative with respect to r gives

$$P'(r) = 4(\tilde{\mathcal{E}}^2 - 1)r^3 + 6Mr^2 + 2[a^2(\tilde{\mathcal{E}}^2 - 1) - \tilde{\mathcal{L}}^2]r + 2M(\tilde{\mathcal{L}} - a\tilde{\mathcal{E}})^2. \tag{40}$$

This system of equations looks a bit complicated – it is 2 equations for 2 unknowns, $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{E}}$. In $(\tilde{\mathcal{L}}, \tilde{\mathcal{E}})$-space, both equations are conic sections centered at the origin, and so they intersect at up to 4 points (2 pairs of points at opposite $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{L}}$). Given a general problem of this type:

$$A_1\tilde{\mathcal{L}}^2 + 2B_1\tilde{\mathcal{L}}\tilde{\mathcal{E}} + C_1\tilde{\mathcal{E}}^2 + F_1 = 0 \quad \text{and} \quad A_2\tilde{\mathcal{L}}^2 + 2B_2\tilde{\mathcal{L}}\tilde{\mathcal{E}} + C_2\tilde{\mathcal{E}}^2 + F_2 = 0, \tag{41}$$

the solution is to take F_2 times the first equation minus F_1 times the second equation, and divide by $\mathcal{E}$, which leads to a quadratic equation for $\tilde{\mathcal{L}}/\tilde{\mathcal{E}}$:

$$(F_2A_1 - F_1A_2)\left(\frac{\tilde{\mathcal{L}}}{\tilde{\mathcal{E}}}\right)^2 + 2(F_2B_1 - F_1B_2)\frac{\tilde{\mathcal{L}}}{\tilde{\mathcal{E}}} + (F_2C_1 - F_1C_2) = 0. \tag{42}$$

We can then solve for $\psi = \tilde{\mathcal{L}}/\tilde{\mathcal{E}}$ using the quadratic formula, and then plug into either equation to find

$$\tilde{\mathcal{E}} = \pm\sqrt{\frac{-F_1}{C_1 + 2B_1\psi + A_1\psi^2}}. \tag{43}$$

If at either stage one encounters the square root of a negative number, there is no solution.

Plots of the boundary regions are shown in Fig. 1.

D. The ISCO

Of particular interest is the ISCO, where $P(r)$ has a triple root at $r_{\rm ISCO}$. In general, if a 4th order polynomial with a root at 0 has a triple root at $r_{\rm ISCO}$, we must have

$$P(r) = ({\rm const})r(r - r_{\rm ISCO})^3 = ({\rm const})(r^4 - 3r_{\rm ISCO}r^3 + 3r_{\rm ISCO}^2 r^2 - r_{\rm ISCO}^3 r), \tag{44}$$

and then from Eq. (39) we find

$$-3r_{\rm ISCO} = \frac{2M}{\tilde{\mathcal{E}}^2 - 1}, \quad 3r_{\rm ISCO}^2 = a^2 - \frac{\tilde{\mathcal{L}}^2}{\tilde{\mathcal{E}}^2 - 1}, \quad \text{and} \quad -r_{\rm ISCO}^3 = \frac{2M(\tilde{\mathcal{L}} - a\tilde{\mathcal{E}})^2}{\tilde{\mathcal{E}}^2 - 1}. \tag{45}$$

This is a system of 3 equations for 3 unknowns: $\tilde{\mathcal{L}}$, $\tilde{\mathcal{E}}$, and $r_{\rm ISCO}$. It can be solved by the more-or-less standard method of substitution. The first equation allows us to write $\tilde{\mathcal{E}}$ in terms of $r_{\rm ISCO}$:

$$\frac{1}{\tilde{\mathcal{E}}^2 - 1} = -\frac{3r_{\rm ISCO}}{2M} \quad \rightarrow \quad \tilde{\mathcal{E}} = \sqrt{1 - \frac{2M}{3r_{\rm ISCO}}}. \tag{46}$$

(There is a negative-energy root, but it turns out to give a particle going backward in time.) The second equation then gives

$$3r_{\rm ISCO}^2 = a^2 + \frac{3r_{\rm ISCO}}{2M}\tilde{\mathcal{L}}^2 \quad \rightarrow \quad \tilde{\mathcal{L}} = \sqrt{2Mr_{\rm ISCO} - \frac{2Ma^2}{3r_{\rm ISCO}}}. \tag{47}$$

(I chose the positive root for a prograde orbit.) The last equation gives

$$\tilde{\mathcal{L}} - a\tilde{\mathcal{E}} = \frac{1}{\sqrt{3}} r_{\rm ISCO} \tag{48}$$

(again I take the branch corresponding to the prograde solution), or

$$\sqrt{2Mr_{\rm ISCO} - \frac{2Ma^2}{3r_{\rm ISCO}}} - a\sqrt{1 - \frac{2M}{3r_{\rm ISCO}}} = \frac{1}{\sqrt{3}} r_{\rm ISCO}. \tag{49}$$

Rearranging the radicals and squaring twice can turn this into a quartic equation for $r_{\rm ISCO}$, and there is a solution by radicals. However, it is most convenient to see what happens to the ISCO by inspection: the solution is at $r_{\rm ISCO} = 6M$ for $a = 0$, and goes to M for $a/M = 1$ (maximum rotation, prograde) and to $9M$ for $a/M = -1$ (maximum rotation, retrograde). It can be seen that the ISCO dips deeper into the hole's potential well for prograde rotation. In the limit of a maximally rotating hole, in the prograde direction (as one might expect if the hole has grown mainly from an accretion disc) then we have

$$\tilde{\mathcal{E}} \rightarrow \frac{1}{\sqrt{3}} = 0.577. \tag{50}$$

Thus for a maximally rotating hole, there are stable orbits down to an energy of $0.577\mu c^2$, so an amount of energy up to $0.423\mu c^2$ can be extracted by frictional forces in a stable disc. This means that an accretion disc around a rotating hole may be much more efficient in converting the mass of infalling matter into radiated energy – a fact of key importance for the theory of active galactic nuclei.

IV. HORIZON PROPERTIES

Finally, we turn our attention to the properties of the outer horizon in the Kerr metric. These are of key importance to black hole thermodynamics, which we will treat next. One such property – the fact that there is a single solid-body rotation speed $\Omega_{\rm H}$ for the horizon – has already been discussed.

A. The horizon area

If we take a slice at the outer event horizon, $r = r_+$ (so $\Delta = 0$), then the metric expressed in terms of θ and ϕ gives an area element

$$\sqrt{g_{\theta\theta}g_{\phi\phi} - (g_{\theta\phi})^2}\,\mathrm{d}\theta\,\mathrm{d}\phi = (r_+^2 + a^2)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi, \tag{51}$$

so the horizon area is

$$A = (r_+^2 + a^2)\int \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = 4\pi(r_+^2 + a^2) = 8\pi\left(M^2 + M\sqrt{M^2 - a^2}\right). \tag{52}$$

For the special case of the Schwarzschild black hole, we have $A = 16\pi M^2$. We will show in the next lecture that the horizon area can never decrease.

B. The thermodynamic identity for rotation

If you think back to your thermodynamics class, you learned that there was a thermodynamic identity for the total energy (here written M, since in relativity I want to identify that with mass):

$$\mathrm{d}M = T\,\mathrm{d}S - P\,\mathrm{d}V + \ldots, \tag{53}$$

where for each globally conserved quantity (e.g., volume V) there is a conjugate variable (here pressure P) that is equal for all components in thermodynamic equilibrium. You don't usually think about it in thermodynamics, but angular momentum J is also conserved, and there is a conjugate variable (angular velocity Ω: this is the same for all objects if they achieve equilibrium with each other after dissipating differential rotation due to friction) with

$$\mathrm{d}M = T\,\mathrm{d}S + \Omega\,\mathrm{d}J. \tag{54}$$

(There's not a useful notion of conserved volume in the case of black holes.) For an object with no internal degrees of freedom (so no S), we have $M = M_0 + \frac{1}{2}I\Omega^2$ and $J = I\Omega$, where I is the moment of inertia. You can easily check Eq. (54) in that case, but the idea is much more general, depending only on the existence of a globally conserved J. It gives a somewhat interesting thermodynamic definition of angular velocity, $\Omega = (\partial M/\partial J)_S$, where the partial derivative is taken along an adiabatic direction.

A stationary black hole is certainly an object that is internally in equilibrium with itself, so you might wonder if something like Eq. (54) applies to black holes. In fact, it does.

We can see how part of this works by re-writing the area formula as

$$A = 8\pi\left(M^2 + \sqrt{M^4 - J^2}\right). \tag{55}$$

Then

$$\mathrm{d}A = 16\pi\left(M + \frac{M^3}{\sqrt{M^4 - J^2}}\right)\mathrm{d}M - 8\pi\frac{J}{\sqrt{M^4 - J^2}}\,\mathrm{d}J. \tag{56}$$

Now we can find $(\partial M/\partial J)_A$ by setting $\mathrm{d}A = 0$:

$$\left(\frac{\partial M}{\partial J}\right)_A = \frac{8\pi J/\sqrt{M^4 - J^2}}{16\pi(M + M^3/\sqrt{M^4 - J^2})} = \frac{J}{2(M\sqrt{M^4 - J^2} + M^3)} = \frac{a}{2M(\sqrt{M^2 - a^2} + M)} = \Omega_{\mathrm{H}}. \tag{57}$$

Thus the horizon of a black hole looks like a thermodynamic system. The only difference is that instead of taking an adiabatic derivative with at constant S, we used constant A. The success of this suggests that the horizon area is related to some kind of entropy; since both are extensive quantities, we suspect that the area A is actually related to the entropy S by some proportionality constant. Nobody has ever measured the entropy of a black hole, but it turns out that this entropy and the associated "temperature" T are predicted by quantum field theory in curved spacetime. Moreover, as a purely classical GR statement, the area of black hole horizons cannot decrease, just like the entropy S in thermodynamics. We will take the next couple of lectures to flesh out these ideas.

Appendix A: Code for circular orbits

Here is the script for `kerr.py`, which prints a table of r/M, $\tilde{\mathcal{L}}/M$, $\tilde{\mathcal{E}}$, $\mathrm{d}t/\mathrm{d}\tau$, and $\Omega = \mathrm{d}\phi/\mathrm{d}t$ for circular equatorial orbits in the Kerr spacetime.

```
import numpy
import sys

# This program works in units where M=1
# and prints the positive (prograde, forward in time) branch of the circular orbits.
#
# Run with negative a to get the retrograde orbits.
a = float(sys.argv[1])

r = 10.**2.3
good = 1
while good:
  # get coefs from notes
  A1 = -r**2+2*r
  B1 = -2*a*r
  C1 = r**4 + a**2*r**2 +2*r*a**2
  F1 = -r**4 + 2*r**3 -a**2*r**2
  A2 = -2*r+2
  B2 = -2*a
  C2 = 4*r**3 + 2*a**2*r +2*a**2
  F2 = -4*r**3 + 6*r**2 -2*a**2*r
  #
  # quadratic equation for psi
  ap = F2*A1-F1*A2
  bp = 2*(F2*B1-F1*B2)
  cp = F2*C1-F1*C2
  #
  #
  discr = bp**2-4*ap*cp
  if discr>=0:
    psi = (-bp + numpy.sqrt(discr))/(2*ap) # since ap>0, this is larger branch
    invet2 = -(C1+2*B1*psi+A1*psi**2)/F1
    if invet2>0:
      Etilde = invet2**(-.5)
      Ltilde = psi*Etilde
      #
      # get inverse metric to raise indices
      Delta = r**2-2*r+a**2
      ginvtt = -((r**2+a**2)**2-a**2*Delta)/Delta/r**2
      ginvtp = -2*a/r/Delta
      ginvpp = (Delta-a**2)/Delta/r**2
      tdot = ginvtp * Ltilde - ginvtt * Etilde
      pdot = ginvpp * Ltilde - ginvtp * Etilde
      print ('{:12.5E} {:12.5E} {:12.5E} {:12.5E} {:12.5E}'.format(r,Ltilde,Etilde,
        tdot,pdot/tdot))
    else:
      good = 0
  else:
    good = 0
  r *= 10**(-.0001)
```

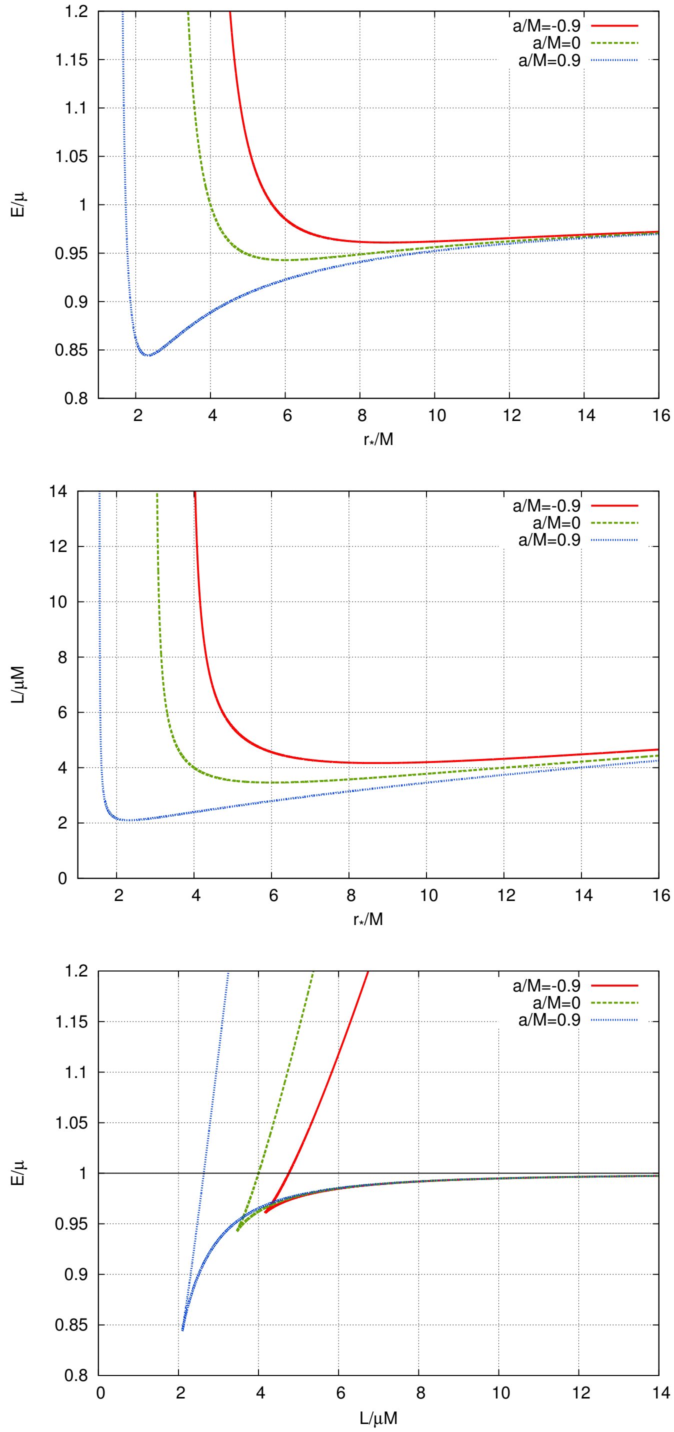

FIG. 1: Circular equatorial orbits in Kerr. *Top panel*: The specific energy $\tilde{\mathcal{E}}$ vs. r. *Middle panel*: The specific angular momentum $\tilde{\mathcal{L}}$ vs. r. *Bottom panel*: Phase diagram showing the boundaries of cases A, B, C, and D in the $(\tilde{\mathcal{L}}, \tilde{\mathcal{E}})$-plane. Note that I considered prograde orbits; $a/M < 0$ is used in this diagram to indicate orbits that are retrograde with respect to the hole's rotation.