# Lecture XVIII: The nature of the event horizon

(Dated: November 1, 2019)

## I. OVERVIEW

Having discussed the behavior of orbits in the Schwarzschild metric,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{1}$$

we now turn our attention to the problem of what happens at $r = 2M$. The equations as written break down, but it will turn out that this is just a problem with the coordinate system in much the same way that the North and South Poles are a problem with latitude and longitude (but there is no catastrophe that happens to someone who reaches these points).

## II. THE RADIALLY INFALLING OBSERVER

Before we reach $r = 2M$, let's begin by asking what happens to an observer $\mathcal{O}$ who free-falls radially inward ($\tilde{\mathcal{L}} = 0$) but from $\infty$ so that $\tilde{\mathcal{E}} = 1$. (The last condition is not strictly necessary for any of our conclusions, but it simplifies the algebra.) We will work at $r > 2M$ but study the limit as $r \to 2M$. I will often write

$$r = 2M(1 + \epsilon), \tag{2}$$

where $\epsilon$ is a small positive number.

### A. The proper time

Let's ask what happens to $\mathcal{O}$'s proper time as they approach $2M$. We first recall from Lecture XVII that

$$\left(\frac{dr}{d\tau}\right)^2 = (u^r)^2 = \tilde{\mathcal{E}}^2 - \left(1 + \frac{\tilde{\mathcal{L}}^2}{r^2}\right)\left(1 - \frac{2M}{r}\right) = \frac{2M}{r} \tag{3}$$

for $\tilde{\mathcal{L}} = 0$ and $\tilde{\mathcal{E}} = 1$. We take the negative root, $dr/d\tau < 0$, since our observer is falling inward. Solving for $\tau$ gives

$$\tau = \int -\frac{dr}{\sqrt{2M/r}} = \text{constant} - \frac{1}{3}\sqrt{\frac{2}{M}}\, r^{3/2}. \tag{4}$$

One can see that $\mathcal{O}$ reaches $r = 2M$ and even $r = 0$ at finite proper time. Let's set the constant so that $\tau = 0$ when the observer reaches $r = 2M$; then

$$\tau = -\frac{1}{3}\sqrt{\frac{2}{M}}\,[r^{3/2} - (2M)^{3/2}] = -\frac{4}{3}M\,[(1+\epsilon)^{3/2} - 1] \approx -2M\epsilon, \tag{5}$$

where the last approximation is for $\epsilon \ll 1$.

### B. The coordinate time

Something very different happens to the coordinate time along observer $\mathcal{O}$'s world line. In particular,

$$\frac{dr}{dt} = \frac{u^r}{u^t} = \frac{\sqrt{2M/r}}{g^{tt}u_t} = \frac{-\sqrt{2M/r}}{-(1 - 2M/r)^{-1}(-1)} = -\left(1 - \frac{2M}{r}\right)\sqrt{\frac{2M}{r}}. \tag{6}$$

This means

$$t = -\int \frac{1}{(1 - 2M/r)\sqrt{2M/r}}\, dr \approx -\int \frac{1}{\epsilon}\, 2M d\epsilon = t_0 - 2M \ln\epsilon, \tag{7}$$

where $t_0$ is an integration constant. We can see that as $\epsilon \to 0^+$, $t \to \infty$. Thus it takes an infinite coordinate time for $\mathcal{O}$ to reach $r = 2M$. But of course that is just a coordinate time; it doesn't tell us what a distant observer sees.

### C. The signals sent to a distant observer

Let's imagine an observer $\mathcal{A}$ looking straight down at the same latitude and longitude as $\mathcal{O}$, but from very far away (i.e., at some $r_\mathcal{A} \gg 2M$). We will suppose that $\mathcal{O}$ sends out radio signals, time-stamped with their proper time $\tau_\mathcal{O}$; we want to know when these signals arrive at $\mathcal{A}$. For a photon, we have $g_{\alpha\beta}(\mathrm{d}x^\alpha/\mathrm{d}t)(\mathrm{d}x^\beta/\mathrm{d}t) = 0$. If the photon moves radially outward, this means

$$-\left(1-\frac{2M}{r}\right)+\frac{1}{1-2M/r}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2=0. \quad\to\quad \frac{\mathrm{d}r}{\mathrm{d}t}=1-\frac{2M}{r}. \tag{8}$$

We can then separate variables and integrate to get

$$t=\int\frac{\mathrm{d}r}{1-2M/r}=r+2M\ln\frac{r-2M}{2M}+\text{constant}. \tag{9}$$

We define the new radial coordinate

$$\bar{r}\equiv r+2M\ln\frac{r-2M}{2M}, \tag{10}$$

which has $\bar{r}\approx r$ for large $r$ but $\bar{r}\approx 2M+2M\ln\epsilon$ for $r\to 2M^+$. Now an outward-going photon moves at constant $t-\bar{r}$. If it is emitted from $\mathcal{O}$ at some proper time $\tau_\mathcal{O}$, then at that time it had:

$$\epsilon\approx\frac{-\tau_\mathcal{O}}{2M},\quad \bar{r}_\mathcal{O}\approx 2M+2M\ln\epsilon,\quad t_\mathcal{O}\approx t_0-2M\ln\epsilon. \tag{11}$$

Now the arrival time at the distant observer $\mathcal{A}$ is $t_\mathcal{O}-\bar{r}_\mathcal{O}+\bar{r}_\mathcal{A}$, or

$$t_\mathcal{A}=t_\mathcal{O}-\bar{r}_\mathcal{O}+\bar{r}_\mathcal{A}\approx(t_0-2M\ln\epsilon)-(2M+2M\ln\epsilon)+\bar{r}_\mathcal{A}=t_0+\bar{r}_\mathcal{A}-2M-4M\ln\epsilon=t_0+\bar{r}_\mathcal{A}-2M-4M\ln\frac{-\tau_\mathcal{O}}{2M}. \tag{12}$$

Thus as $\mathcal{O}$ plunges toward $r=2M$, signals take longer and longer to reach $\mathcal{A}$. The signal emitted at $\tau_\mathcal{O}=0$ ($r=2M$) never reaches $\mathcal{A}$. Thus the surface $r=2M$ is known as the *event horizon*: even light cannot get from that surface to a distant observer!

Now $\mathcal{O}$ reaches the event horizon in a finite amount of proper time, but the whole process takes infinitely long as seen by a distant observer. That distant observer, looking at the black hole, sees everything that has fallen into it "frozen" at the instant those objects crossed the horizon. That includes the star that collapsed to make the black hole, assuming it has a finite age (as all real black holes do).

But of course it is hard to truly make this observation: the signals sent out by observer $\mathcal{O}$ are redshifted:

$$1+z=\frac{\mathrm{d}t_\mathcal{A}}{\mathrm{d}\tau_\mathcal{O}}\approx\frac{2M}{-\tau_\mathcal{O}}\approx\text{const}\times\mathrm{e}^{t_\mathcal{A}/4M}. \tag{13}$$

The redshift increases exponentially, with an e-folding timescale of $4M=20(M/M_\odot)\,\mu$s. So after a few e-folds, the apparent luminosity of the objects that have fallen into the black hole can be neglected for all practical purposes; a black hole looks black.

### D. Signals seen by the radially infalling observer

We may also ask what $\mathcal{O}$ sees of the distant Universe. If $\mathcal{O}$ crosses the horizon at finite proper time, do they notice anything as they do so? It turns out the answer is no.

We might first ask how $\mathcal{O}$ perceives the observer $\mathcal{A}$, who we will assume has helpfully shined a laser beam directly down at $\mathcal{O}$ as they fall into the hole. The beam of photons has energy $E_\mathcal{A}$ at $\infty$ (or as emitted by $\mathcal{A}$, if $r_\mathcal{A}\gg 2M$). Then for the photon beam, $p_t=-E_\mathcal{A}$. The null condition $g^{\alpha\beta}p_\alpha p_\beta=0$ then implies

$$-\frac{1}{1-2M/r}(-E_\mathcal{A})^2+\left(1-\frac{2M}{r}\right)(p_r)^2=0\quad\to\quad p_r=-\frac{1}{1-2M/r}E_\mathcal{A}. \tag{14}$$

Now the energy seen by $\mathcal{O}$ is related to the photon 4-momentum $\boldsymbol{p}$ and the observer's 4-velocity $\boldsymbol{u}$ by

$$E_\mathcal{O}=-p_\alpha u^\alpha=-p_t u^t-p_r u^r=E_\mathcal{A}\frac{1}{1-2M/r}-\left(-\frac{1}{1-2M/r}E_\mathcal{A}\right)\left(-\sqrt{\frac{2M}{r}}\right)=\frac{1-\sqrt{2M/r}}{1-2M/r}E_\mathcal{A}. \tag{15}$$

Now we are fortunate that as $r \to 2M$:

$$\frac{1-\sqrt{2M/r}}{1-2M/r} = \frac{1-(1+\epsilon)^{-1/2}}{1-(1+\epsilon)^{-1}} \approx \frac{1}{2}. \tag{16}$$

Thus $\mathcal{O}$ sees a redder wavelength than emitted by $\mathcal{A}$, but even as $\mathcal{O}$ approaches the horizon, they only see the photon energy shifted by a factor of 2 (i.e., total redshift $1+z=2$).

### E. The appearance of the sky

We may also ask what the sky looks like to an observer right above the horizon. We will address this by placing an observer $\mathcal{B}$ at the same spacetime event as $\mathcal{O}$, but stationary with respect to the hole – so the 4-velocity of $\mathcal{B}$ is $((1-2M/r)^{-1/2}, 0, 0, 0)$. We will work at small $\epsilon$, so inside the photon sphere ($r < 3M$).

Because of the extraordinary bending of light, $\mathcal{B}$ sees themselves surrounded by the black hole. Only photons with $\mathcal{L}/\mathcal{E} < 3\sqrt{3}\,M$ can penetrate from $\infty$ into the inner region (i.e., with no forbidden radial region). If we put the observer at the equator, and consider photon trajectories in the equatorial plane, this means that

$$\frac{p_\phi}{-p_t} < 3\sqrt{3}\,M. \tag{17}$$

Written in terms of the local orthonormal coordinates at $\mathcal{B}$,

$$\frac{p_{\hat{\phi}}}{-p_{\hat{t}}} = \frac{p_\phi/r}{-p_t/\sqrt{1-2M/r}} < 3\sqrt{3}\sqrt{1-\frac{2M}{r}}\frac{M}{r}. \tag{18}$$

This is the ratio of the horizontal momentum to the energy (or total momentum). Thus if observer $\mathcal{B}$ looks up, they see sky only in a region of opening angle $\alpha$, where

$$\sin\alpha = 3\sqrt{3}\sqrt{1-\frac{2M}{r}}\frac{M}{r}. \tag{19}$$

Everywhere else they see the black hole. We see that at the photon sphere, $\alpha = \frac{\pi}{2}$, which makes sense: photons can orbit the hole, so $\mathcal{B}$ sees the black hole below them and the sky above; if they look perfectly horizontally they see themselves. Outside the photon sphere, $\alpha > \frac{\pi}{2}$ (the empty sky fills more than half of $\mathcal{B}$'s view). As $\mathcal{B}$ moves down toward the horizon, we have

$$\alpha \to \frac{3\sqrt{3}}{2}\sqrt{\epsilon}. \tag{20}$$

So just above the horizon, $\mathcal{B}$ would see just a tiny bit of sky straight above them, highly blueshifted, and a vast expanse of black hole surrounding them, covering $\approx 4\pi - \pi\alpha^2$ steradians. In fact, $\mathcal{B}$ can see the whole sky: actually $\mathcal{B}$ sees infinitely many images of every star in the sky, because they see photons that can orbit around the light sphere $n$ times before plunging in, where $n$ is any positive integer.

The infalling observer $\mathcal{O}$ sees something totally different. They have a velocity in $\mathcal{B}$'s frame of

$$v_{\mathcal{OB}} = \frac{u^{\hat{r}}}{u^{\hat{t}}} = \frac{1}{1-\frac{2M}{r}}\frac{u^r}{u^t} = -\sqrt{\frac{2M}{r}} = -\frac{1}{\sqrt{1+\epsilon}}. \tag{21}$$

Now the opening angle $\alpha_{\mathcal{O}}$ as seen by $\mathcal{O}$ is related to this by the usual special relativistic transform (recall $\mathcal{O}$ and $\mathcal{B}$ are instantaneously at the same place!). Using $\cos\alpha \approx 1 - \frac{1}{2}\alpha^2$, we find

$$\begin{aligned}
\cos\alpha_{\mathcal{O}} &= \frac{v_{\mathcal{OB}} + \cos\alpha}{1 + v_{\mathcal{OB}}\cos\alpha} \\
&= \frac{-(1+\epsilon)^{-1/2} + (1-\frac{27}{8}\epsilon)}{1-(1+\epsilon)^{-1/2}(1-\frac{27}{8}\epsilon)} \\
&\approx \frac{-1+\frac{1}{2}\epsilon + 1 - \frac{27}{8}\epsilon}{1-(1-\frac{1}{2}\epsilon)(1-\frac{27}{8}\epsilon)} \\
&\approx \frac{(\frac{1}{2}-\frac{27}{8})\epsilon}{(\frac{1}{2}+\frac{27}{8})\epsilon} = -\frac{23}{31} \quad\to\quad \alpha_{\mathcal{O}} \approx 138^\circ.
\end{aligned} \tag{22}$$

Thus the observer falling into the hole from rest at $\infty$ sees the hole first appear small, then grow. When $\mathcal{O}$ reaches the horizon the black hole appears to take up an angular radius of 42° and the outside universe takes up an angular radius of 138°. Nothing special happens right at the horizon! It may seem odd that $\mathcal{O}$ can see light from the outside universe appear in their "downward-facing" hemisphere, but these are just photons with some transverse motion such that the infalling observer catches up with them.

## III. CONTINUING PAST $r = 2M$

To understand what really happens at $r = 2M$, it is helpful to change coordinates to eliminate the coordinate singularity. The price to pay is that the time translation invariance of the Schwarzschild solution is no longer going to be self-evident. We have already defined $\bar{r}$ by Eq. (10), which is related to $r$. (We can't write $r$ in terms of $\bar{r}$ in closed form, but recall $\bar{r} \approx r$ for large $r$ and $\bar{r} \to -\infty$ for $r \to 2M$.) This coordinate choice allows us to write

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r}\right)(-\mathrm{d}t^2 + \mathrm{d}\bar{r}^2) + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{23}$$

Now let's rotate the coordinate system as follows –

$$\bar{\xi} = t + \bar{r} \quad \text{and} \quad \bar{\eta} = t - \bar{r}, \tag{24}$$

so now $\bar{r}$ (and hence $r$) are increasing functions of $\bar{\xi} - \bar{\eta}$. We have

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}\bar{\xi}\,\mathrm{d}\bar{\eta} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{25}$$

Here the valid range of $(\bar{\xi}, \bar{\eta})$ is the whole of $\mathbb{R}^2$ (this corresponds to $r > 2M$, $-\infty < t < \infty$). Particles going "forward" in time must be moving to the upper-right in $(\bar{\xi}, \bar{\eta})$-space. Radially traveling light rays ($\mathrm{d}\theta/\mathrm{d}\lambda = \mathrm{d}\phi/\mathrm{d}\lambda = 0$) have null trajectories and so move straight up (ingoing) or straight right (outgoing).

So far, we haven't actually resolved the coordinate singularity at $r = 2M$, i.e., $\bar{r} = -\infty$: we have placed it in the far upper-left quadrant of $(\bar{\xi}, \bar{\eta})$-space. Another coordinate transformation is necessary to do this; let's define

$$\xi = \mathrm{e}^{\bar{\xi}/4M}, \quad \eta = -\mathrm{e}^{-\bar{\eta}/4M} \qquad \leftrightarrow \qquad \bar{\xi} = 4M\ln\xi, \quad \bar{\eta} = -4M\ln\eta. \tag{26}$$

In this case,

$$\bar{r} = \frac{1}{2}(\bar{\xi} - \bar{\eta}) = 2M\ln(-\xi\eta) \quad \to \quad \xi\eta = -\mathrm{e}^{\bar{r}/2M} = -\frac{r - 2M}{2M}\mathrm{e}^{r/2M}. \tag{27}$$

We can then write

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r}\right)\frac{16M^2\,\mathrm{d}\xi\,\mathrm{d}\eta}{\xi\eta} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) = \frac{32M^3}{r}\mathrm{e}^{-r/2M}\,\mathrm{d}\xi\,\mathrm{d}\eta + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{28}$$

In the $(\xi, \eta, \theta, \phi)$ coordinate system, the event horizon $r = 2M$ corresponds to $\xi\eta = 0$, i.e., to both of the axes in the $(\xi, \eta)$-plane. The region $r > 2M$ – the exterior to the black hole – is in the lower-right quadrant (Quadrant IV). From Eq. (27), we can see that $r = 0$ at $\xi\eta = 1$: these two hyperbolas (in Quadrants I and III) are true singularities (one can check that, e.g., $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ blows up there). But at $\xi\eta < 1$, the metric of Eq. (28) is well-behaved. In particular, as particles move into the future and cross the horizon (Quadrant IV→I: $\xi > 0$, $\eta = 0$) nothing special happens. But because of the structure of the metric, once this horizon is crossed, there is no way back out.

Note that in this picture, time translation invariance is a rescaling of $\xi$ and $\eta$: $t \to t + \Delta$ is equivalent to $\xi \to \mathrm{e}^{\Delta/4M}\xi$ and $\eta \to \mathrm{e}^{-\Delta/4M}\eta$, but with $\xi\eta$ (and hence $r$) being fixed.

If Quadrant IV is "normal spacetime" and Quadrant I is the interior of the black hole, you might wonder about Quadrants II and III. In particular, a particle can emerge from Quadrant III and enter Quadrant IV, thus appearing out of a past horizon at $t = -\infty$ and entering our visible universe. But remember Eq. (28) was derived by an analytic continuation. The Schwarzschild metric is the metric exterior to a spherical star (even if dynamic, though I haven't proven this), but not interior to it. So if we try to follow particle paths back to Quadrant III, we inevitably hit the surface of the star that collapsed to make the black hole. So Quadrant III, and similarly Quadrant II, are not of physical relevance.

What is the fate of a particle that crosses the horizon and enters Quadrant I? Unfortunately, it is destined to reach $r = 0$ (and as we saw earlier, a freely falling particle does so in finite proper time). As the curvature (hence tidal field) increases, any extended particle must be shredded. For a truly elementary particle, its fate must be determined by some kind of breakdown of general relativity as one approaches infinite curvature. Maybe this is quantum gravity if the curvature scale $(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta})^{-1/4}$ reaches the Planck length.

*Comment* – the book rotates the coordinates again, and defines

$$u = \frac{\xi - \eta}{2} \quad \text{and} \quad v = \frac{\xi + \eta}{2}. \tag{29}$$

This rotates everything back to the usual orientation, where particles must move within 45° of straight up.