Lecture XVII: Geodesics in the Schwarzschild geometry

(Dated: October 30, 2019)

I. OVERVIEW

We now consider the Schwarzschild metric (as defined in Lecture XVI), which applies outside of any spherically symmetric star. We will consider the trajectories of test particles (the geodesics) in the Schwarzschild metric. Since the metric is spherically symmetric, we will focus on trajectories in the equatorial plane.

II. THE BASICS, AND THE RADIAL EQUATION

Let's first take the Schwarzschild metric:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-2M/r} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2); \tag{1}$$

we will consider only the region $r > 2M$ here (it is clear that something goes weird with the mathematics at $r = 2M$ – in fact, that is the *event horizon* – but let's defer its physical nature for the moment).

Now suppose that we have a massive test particle. (I will be different from the book and take the massless particle as a limit of mass $\mu \to 0$ at the end.) The metric does not depend on t or ϕ, and so the momenta p_t and p_ϕ are both conserved; we write the particle's energy per unit mass $\tilde{\mathcal{E}} = -p_t/\mu = -u_t$ and its angular momentum per unit mass $\tilde{\mathcal{L}} = p_\phi/\mu = u_\phi$.

Now for an equatorial orbit, $\theta = \frac{\pi}{2}$ and $u^\theta = 0$. At this stage, we can use the fact that the metric is diagonal to see that

$$u^t = \frac{u_t}{g_{tt}} = \frac{\tilde{\mathcal{E}}}{1-2M/r} \quad \text{and} \quad u^\phi = \frac{u_\phi}{g_{\phi\phi}} = \frac{\tilde{\mathcal{L}}}{r^2}. \tag{2}$$

We only consider particles moving in the forward light cone, so $u^t > 0$ and then $\tilde{\mathcal{E}} > 0$. (Note this is a statement about the Schwarzschild metric; other metrics, including that of a rotating black hole, can have particles moving forward in time but with negative energy.) Furthermore, I will focus on prograde particles with $\tilde{\mathcal{L}} \geq 0$; there is no difference between prograde and retrograde in a spherical solution. (This will also be modified for rotating holes.)

We can then use the normalization of the 4-velocity, $g_{\alpha\beta}u^\alpha u^\beta = -1$, to see that

$$-\frac{\tilde{\mathcal{E}}^2}{1-2M/r} + \frac{(u^r)^2}{1-2M/r} + \frac{\tilde{\mathcal{L}}^2}{r^2} = -1. \tag{3}$$

Rearranging gives

$$(u^r)^2 = \tilde{\mathcal{E}}^2 - \left(1 + \frac{\tilde{\mathcal{L}}^2}{r^2}\right)\left(1 - \frac{2M}{r}\right). \tag{4}$$

This is the radial equation of motion in the Schwarzschild geometry. The right-hand side is a cubic polynomial in $x = 1/r$, which we may write as $P(x)$. Particle motion is only possible in cases where $P(x) \geq 0$; a point with $P(x) = 0$ is a *turning point*, and if the particle reaches there it will have $\mathrm{d}r/\mathrm{d}\tau = u^r = 0$ and then go back into the $P(x) > 0$ region. For each value of $\mathcal{E}$ and $\mathcal{L}$, we must make an assessment of where the turning points are. This will determine whether particles can stably orbit in the Schwarzschild spacetime, or if they go out to $r = \infty$ ($x = 0$), or if they go in to $r = 2M$ ($x = \frac{1}{2M}$).

III. CLASSIFICATION OF THE TURNING POINTS

Let's first make note of a few properties of the polynomial $P(x)$. I will go ahead and expand it as:

$$(u^r)^2 = P(x) = \tilde{\mathcal{E}}^2 - 1 + 2Mx - \tilde{\mathcal{L}}^2x^2 + 2M\tilde{\mathcal{L}}^2x^3. \tag{5}$$

A few properties are immediately clear:

- There are at most 3 turning points. (A polynomial of 3rd degree has 3 roots, and at most 3 are in the range $0 < x < \frac{1}{2M}$.)
- The event horizon $x = \frac{1}{2M}$ is in an allowed region for any $\mathcal{E}$.
- $r = \infty$ or $x = 0$ is an allowed region if and only if $\mathcal{E} \geq 1$.
- If a value of x is allowed for some $\mathcal{L}$ and $\mathcal{E}$, then it is still allowed if $\mathcal{E}$ is increased or $\mathcal{L}$ is decreased.

This in turn implies a finite number of possible cases:

- Case A: $0 < \mathcal{E} < 1$, and there is one root of $P(x)$ between 0 and $\frac{1}{2M}$. There is one allowed region at $x > x_+$.
- Case B: $0 < \mathcal{E} < 1$, and there are three roots of $P(x)$ between 0 and $\frac{1}{2M}$. There are two separate allowed regions at $x > x_+$ (the largest root) and between $x_1 < x < x_2$ (the other roots).
- Case C: $\mathcal{E} > 1$, and there are zero roots of $P(x)$ between 0 and $\frac{1}{2M}$. The entire range is allowed.
- Case D: $\mathcal{E} > 1$, and there are two roots of $P(x)$ between 0 and $\frac{1}{2M}$. There are two separate allowed regions at $x > x_+$ (the larger root) and $x < x_2$ (the smaller root).

There are also some boundary cases: at $\mathcal{E} = 1$, where a root appears at $x = 0$, one may have a transition between Cases A and C ("AC"), and one can also have a repeated root of the polynomial (boundaries AB or CD). Our main interest here is in figuring out which range of $\mathcal{L}$ and $\mathcal{E}$ correspond to which. Examples of each of these cases are shown in Fig. 1.

It turns out that the simplest way to find these boundaries is to search for a repeated root at some $x_\star$. A repeated root has bot $P(x_\star) = 0$ and $P'(x_\star) = 0$, or

$$0 = 2M - 2\tilde{\mathcal{L}}^2 x_\star + 6M\tilde{\mathcal{L}}^2 x_\star^2 \quad \rightarrow \quad \tilde{\mathcal{L}}^2 = \frac{M}{(1 - 3Mx_\star)x_\star}. \tag{6}$$

The energy can then be determined by setting $P(x_\star) = 0$:

$$\tilde{\mathcal{E}}^2 = \left(1 + \tilde{\mathcal{L}}^2 x_\star^2\right)(1 - 2Mx_\star) = \frac{(1 - 2Mx_\star)^2}{1 - 3Mx_\star}. \tag{7}$$

The resulting curve in the $(\tilde{\mathcal{L}}, \tilde{\mathcal{E}})$-plane is shown in Fig. 2.

IV. THE CIRCULAR ORBITS

A repeated root of $P(x)$ is not just a boundary between two cases: it also corresponds to a circular orbit. A repeated root where $P''(x_\star) < 0$ is a stable circular orbit, because $x = x_\star$ is allowed but any deviation from it is forbidden; a repeated root where $P''(x_\star) > 0$ is an unstable circular orbit, because a small perturbation allows the particle to run away either to larger or smaller x.

A. The sequence of orbits

We can see from Eqs. (6) and (7) that circular orbits should exist as long as $x_\star < \frac{1}{3M}$, i.e., $r_\star > 3M$. Moreover, by taking the second derivative:

$$P''(x) = -2\tilde{\mathcal{L}}^2 + 12M\tilde{\mathcal{L}}^2 x = 12M\tilde{\mathcal{L}}^2 \left(x - \frac{1}{6M}\right), \tag{8}$$

we see that a circular orbit is stable for $x_\star < \frac{1}{6M}$, i.e., $r_\star > 6M$, and unstable for $r_\star < 6M$. It is thus conventional to call the circular orbit at $r_\star = 6M$ the *innermost stable circular orbit* (ISCO). The ISCO has dramatic implications for accretion discs around black holes.

We can also look at the energy and angular momentum from Eqs. (6) and (7). For orbits at very large radius, we find the expansions

$$\tilde{\mathcal{L}} = \sqrt{\frac{M}{(1 - 3Mx_\star)x_\star}} \approx \sqrt{\frac{M}{x_\star}} \approx \sqrt{Mr_\star} \tag{9}$$

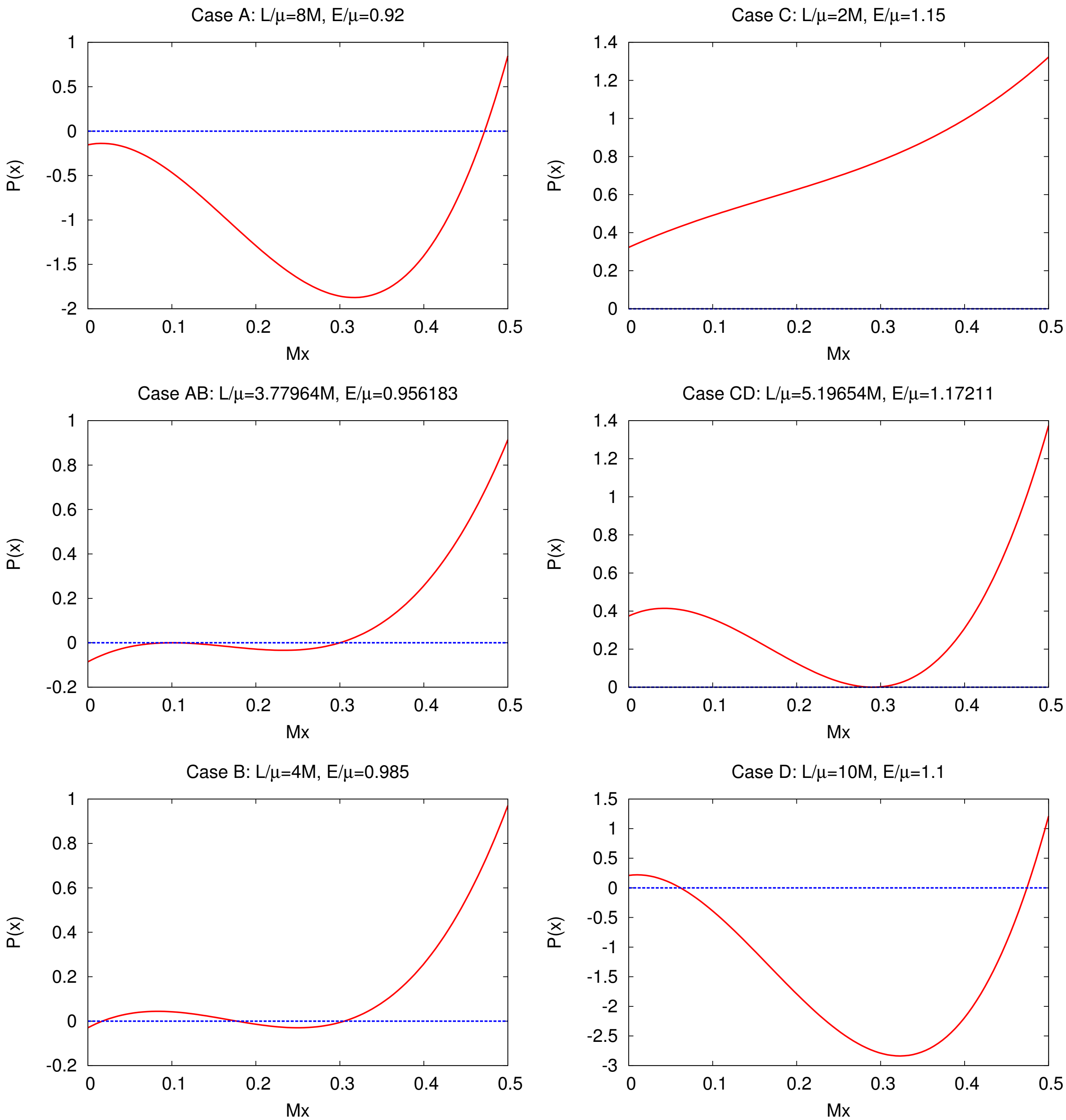

FIG. 1: Some examples of polynomials $P(x)$. Allowed ranges correspond to $P(x) > 0$. Only case B allows stable orbits that neither plunge to $x = \frac{1}{2M}$ nor escape to $x = 0$.

and

$$\tilde{\mathcal{E}} = \frac{1 - 2Mx_\star}{\sqrt{1 - 3Mx_\star}} \approx 1 - \frac{1}{2}Mx_\star = 1 - \frac{M}{2r_\star}, \tag{10}$$

which are the usual Newtonian expressions. However, as the orbit shrinks, it turns out that both $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{E}}$ go through a minimum and increase again, reaching ∞ as $r_\star \to 3M$. One can show by brute force differentiation that both $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{L}}$ have their minimum at $x_\star = \frac{1}{6M}$, i.e., at the ISCO. The values here are

$$\tilde{\mathcal{L}}_{\rm ISCO} = 2\sqrt{3}\,M \approx 3.464M \quad \text{and} \quad \tilde{\mathcal{E}}_{\rm ISCO} = \frac{2\sqrt{2}}{3} \approx 0.943. \tag{11}$$

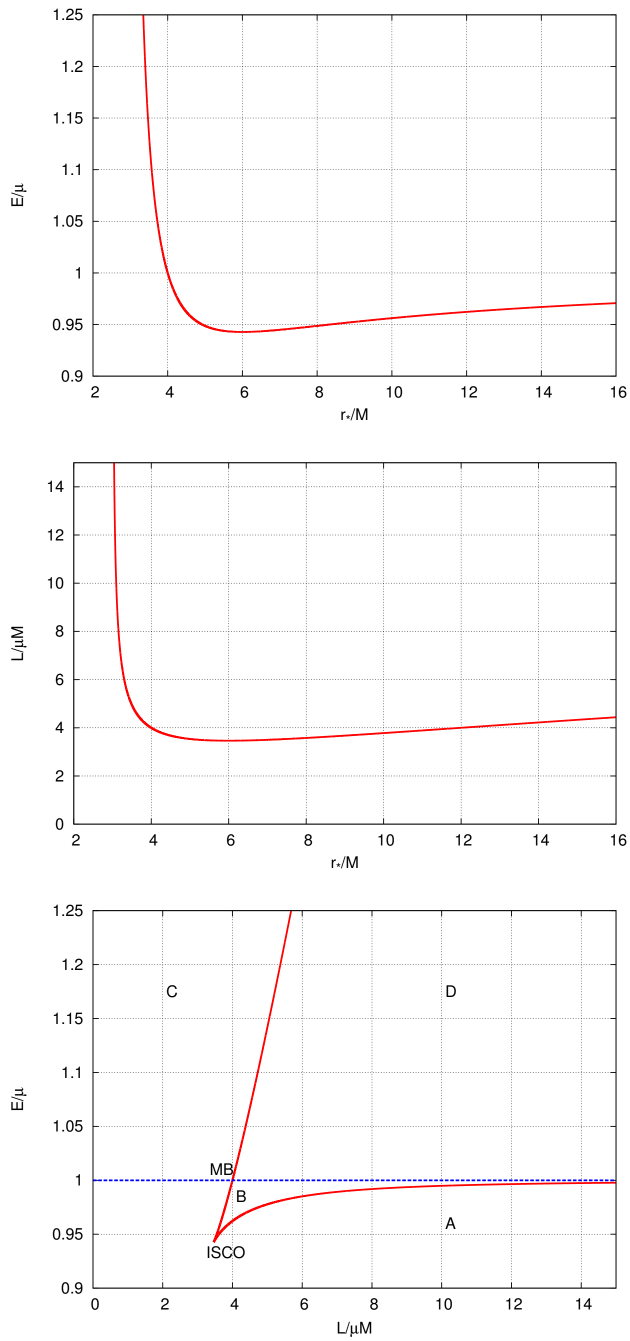

FIG. 2: The curves of $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{L}}$ vs. $r_\star$ (top two panels) and the phase diagram of the $(\tilde{\mathcal{L}}, \tilde{\mathcal{E}})$-plane (bottom panel), with the 4 cases shown. The special points "MB" and "ISCO" are also shown.

The ISCO is the cusp of the AB boundary, and hence is the lowest-energy orbit around the hole that neither plunges to $2M$ nor escapes to ∞.

Circular orbits interior to the ISCO are unstable. However, one case of particular interest is the *marginally bound* (MB) orbit, which occurs when $r_\star = 4M$ and $\tilde{\mathcal{E}} = 1$. For circular orbits outside $4M$, a small outward perturbation causes the particle to fly off but then fall back toward the central mass (there is another turning point) and ultimately plunge into the event horizon; for a circular orbit inside $4M$, a small outward perturbation causes the particle to zoom off to ∞.

The orbit at $r_\star = 3M$, with $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{L}} \to \infty$, is only accessible to massless particles, where the energy and angular momentum remain finite but $\mu \to 0$. This orbit is called the *photon sphere* because photons can orbit the black hole.

The photon sphere is unstable: photons orbiting the hole either escape to ∞ or plunge into the hole.

B. Orbital properties

For the circular orbit, we may compute the 4-velocity components:

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = u^t = \frac{\tilde{\mathcal{E}}}{1-2M/r_\star} = \left(1 - \frac{3M}{r_\star}\right)^{-1/2} \tag{12}$$

and

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = u^\phi = \frac{\tilde{\mathcal{L}}}{r_\star^2} = \frac{1}{r_\star}\sqrt{\frac{M}{r_\star - 3M}}. \tag{13}$$

The first of these equations describes the mean rate of passage of time for an observer at rest at ∞ ($\mathrm{d}t$) versus a clock on a circular orbit ($\mathrm{d}\tau$). Since $\mathrm{d}t/\mathrm{d}\tau > 1$, the observer at ∞ measures the passage of time at a faster rate. The effect is small in the Solar System: since the Earth's orbit around the Sun is at $r/M = 1.0\times 10^8$, we have $\mathrm{d}t/\mathrm{d}\tau - 1 = 1.5\times 10^{-8}$. That is, every year, a clock on the Earth loses 1.5×10^{-8} yr or 0.5 s relative to a clock outside the Solar System. But near a black hole, this is a large effect: at the ISCO, $\mathrm{d}t/\mathrm{d}\tau = \sqrt{2}$.

One can also ask about the orbital period (as measured by an observer at ∞, e.g., us looking at a particle orbiting a black hole). This is $2\pi/\Omega$, where

$$\Omega = \frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{u^\phi}{u^t} = \sqrt{\frac{M}{r_\star^3}}. \tag{14}$$

Amazingly, and coincidentally, this agrees with Kepler's 3rd law with no corrections. But of course, the interpretation of r is different.

V. PRECESSION OF THE ORBIT

One of the important tests of GR is the precession of the orbit of Mercury. (A similar test is now possible with binary stars, but I will focus on a test particle orbiting a spherical mass; Mercury can be considered to have negligible mass compared to the Sun.) We will therefore study the precession of slightly elliptical orbits (i.e., those where r varies only a little), but at arbitrary radius r.

We begin by recalling that $(u^r)^2 = P(x)$. For a circular orbit, we can write – in the vicinity of the double root –

$$P(x) \approx \frac{1}{2}P''(x_\star)(x - x_\star)^2. \tag{15}$$

For an orbit that is slightly elliptical, so the energy is slightly greater than that of the circular orbit such that there is a small $\delta(\tilde{\mathcal{E}}^2)$, we can see that $P(x)$ is increased by $\delta(\tilde{\mathcal{E}}^2)$. Then:

$$\dot{r}^2 = (u^r)^2 = P(x) \approx \delta(\tilde{\mathcal{E}}^2) + \frac{1}{2}P''(x_\star)(x - x_\star)^2. \tag{16}$$

Now if we use $\mathrm{d}r/\mathrm{d}x = -1/x^2$, then we can write the left-hand side as

$$\frac{1}{x_\star^4}\dot{x}^2 \approx \delta(\tilde{\mathcal{E}}^2) + \frac{1}{2}P''(x_\star)(x - x_\star)^2. \tag{17}$$

If we take the derivative with respect to τ and divide by $\dot{x}$, we get

$$\frac{2}{x_\star^4}\ddot{x} \approx P''(x_\star)(x - x_\star), \tag{18}$$

so small perturbations around the circular orbit look like a harmonic oscillator with frequency

$$\omega_x^2 = -\frac{1}{2}x_\star^4 P''(x_\star) = -\frac{1}{2}x_\star^4 \times 12M\tilde{\mathcal{L}}^2\left(x_\star - \frac{1}{6M}\right) = -\frac{1}{2}x_\star^4 \times 12M\frac{M}{(1-3Mx_\star)x_\star}\left(x_\star - \frac{1}{6M}\right) = \frac{Mx_\star^3(1-6Mx_\star)}{(1-3Mx_\star)}. \tag{19}$$

Written in terms of $r_\star$, this is

$$\omega_x = \sqrt{\frac{M(1-6M/r_\star)}{r^3(1-3M/r_\star)}}. \tag{20}$$

Now this is the frequency of radial oscillations in radians per unit proper time (i.e., the period is $\Delta\tau = 2\pi/\omega_x$). What we want right now is the frequency in radians per unit time seen at ∞ (i.e., so $\Delta t = 2\pi/\Omega_r$). This requires that we multiply by $\mathrm{d}\tau/\mathrm{d}t$, i.e., divide by Eq. (12). Then:

$$\Omega_r = \left(1 - \frac{3M}{r_\star}\right)^{1/2} \omega_x = \sqrt{\frac{M(1-6M/r_\star)}{r^3}}. \tag{21}$$

It follows that the frequency of radial oscillations is less than the frequency Ω by a factor of $\sqrt{1-6M/r_\star}$: that is, for every time the particle completes a radial oscillation (going from pericenter to apocenter and back to pericenter), it has completed $(1-6M/r_\star)^{-1/2}$ orbits in the ϕ direction. The longitude spacing between successive pericenters is

$$\Delta\phi = \frac{2\pi}{\sqrt{1-6M/r_\star}}. \tag{22}$$

In the Newtonian limit, where $M/r_\star \ll 1$, we have $\Delta\phi \approx 2\pi$, so the pericenters keep occuring at the same longitude and the orbit is closed (an ellipse). The leading correction is

$$\Delta\phi \approx 2\pi + \frac{6\pi M}{r_\star} + ... ; \tag{23}$$

in the case of Mercury orbiting the Sun, the additional angle is

$$\frac{6\pi M}{r_\star} = \frac{6\pi GM}{c^2 r_\star} = 4.8 \times 10^{-7}\,\text{radian} = 0.010\,\text{arcsec}. \tag{24}$$

Thus after each orbit of the Sun, Mercury's pericenter has a relativistic advance of 0.010 arcsec. (This is on top of the Newtonian perturbations from the other planets, which are larger.) This is a tiny angle, but because Mercury goes around the Sun 410 times per century, this adds up to 41 arcsec/century. This correction is enough to measure.

The precession rate increases if the orbit moves inward. At $r_\star = 8M$, $\Delta\phi$ reaches 4π: at this radius, a slightly eccentric orbit closes on itself again, but it swings around the star twice in longitude each time it comes back to pericenter (2:1 ratio of Ω:Ω_r). As one keeps moving inward, one gets to the 3:1, 4:1, etc. ratios; finally, at $r_\star = r_{\rm ISCO} = 6M$, we have $\Omega_r \to 0$ and the radial oscillations are neutrally stable. Orbits inside of that are unstable ($\Omega_r^2 < 0$).

VI. PHOTON TRAJECTORIES

Photons are distinct from massive particles in that $\mu \to 0$ so $\tilde{\mathcal{E}} \to \infty$. However the ratio $\tilde{\mathcal{L}}/\tilde{\mathcal{E}} = p_\phi/(-p_t)$ remains well-behaved. This is the ratio of angular momentum to energy; for a photon coming from ∞, that is the impact parameter b (recall that at ∞, the momentum of the photon is the energy, and angular momentum is momentum times impact parameter).

Now for a photon coming in toward a black hole, an obvious question is whether the photon returns to ∞ (large b) or plunges into the hole (small b). In fact, one can see that the transition $b_{\rm crit}$ occurs at the boundary of Cases C and D, where the turning points present in Case D disappear. The limit of large $\tilde{\mathcal{E}}$ is obtained by taking $r_\star \to 3M$. We then have

$$b_{\rm crit} = \lim_{r_\star \to 3M^+} \frac{\tilde{\mathcal{L}}}{\tilde{\mathcal{E}}} = \lim_{r_\star \to 3M^+} \frac{\sqrt{Mr_\star/(1-3M/r_\star)}}{(1-2M/r_\star)/\sqrt{1-3M/r_\star}} = 3\sqrt{3}\,M. \tag{25}$$

This means that − although the event horizon is at $r = 2M$ and the photon sphere is at $r = 3M$ − the radius over which the hole can pull in a beam of light is larger, $3\sqrt{3}\,M$. This is also the apparent size of the hole's shadow if it is imaged from a distant observer (as was done for the first time this year, in the case of the hole at the center of M87). Photons shot toward the hole with impact parameter slightly greater than $3\sqrt{3}\,M$ will approach the photon sphere and may orbit many times before escaping; photons shot toward the hole with impact parameter slightly less than $3\sqrt{3}\,M$ will cross the photon sphere and orbit many times before plunging into the hole. In geometric optics, a photon shot toward the hole at exactly $b = 3\sqrt{3}\,M$ will asymptotically approach a circular orbit at $r = 3M$, but of course the probability of a photon being exactly at this impact parameter is infinitesimal (and if you did try to set up that experiment, wave optics would become important).