Lecture XV: Gravitational waves: astrophysical sources and detection

(Dated: October 23, 2019)

I. OVERVIEW

We now want to discuss the possible astrophysical sources of gravitational waves and the methods for measuring them. We begin with a discussion of the signal strength from a binary system, and then consider the technologies for measuring gravitational wave strain.

II. STRAIN FROM A BINARY SYSTEM

Let's consider a binary system with masses m_1 and m_2 in a circular orbit as seen pole-on (this is just for simplicity here so I don't have to do a lot of trig functions, but it turns out to give the largest signal). From the previous lecture, the second derivative of the quadrupole moment is

$$\ddot{\mathbf{Q}} = -2\mu a^2\omega^2 \begin{pmatrix} \cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & -\cos 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} = -2\mu a^2\omega^2 \Re \begin{pmatrix} 1 & -\mathrm{i} & 0 \\ -\mathrm{i} & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathrm{e}^{-2\mathrm{i}\omega t}. \tag{1}$$

For an observer at distance r on the z-axis (so that $\ddot{\mathbf{Q}}$ is entirely transverse and traceless), this results in a TT-gauge strain of

$$\mathbf{h}^{\mathrm{TT}} = -\frac{4}{r}\mu a^2\omega^2 \Re \begin{pmatrix} 1 & -\mathrm{i} & 0 \\ -\mathrm{i} & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathrm{e}^{2\mathrm{i}\omega(r-t)}, \tag{2}$$

so

$$h_+ = -\frac{2\mu a^2\omega^2}{r}\cos 2\omega(r-t), \quad h_\times = -\frac{2\mu a^2\omega^2}{r}\sin 2\omega(r-t), \tag{3}$$

where we set

$$h_{\mu\nu} \to \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 2h_+ & 2h_\times & 0 \\ 0 & 2h_\times & -2h_+ & 0 \\ 0 & 0 & 0 & 0 \end{array}\right). \tag{4}$$

(We will talk about what h_+ and $h_\times$ physically mean when we get to gravitational wave detection.)

Let's now put the factors of G and c back into the strain formula,

$$h = \sqrt{h_+^2 + h_\times^2} = \frac{\mu a^2\omega^2}{2r} = \frac{2G\mu a^2\omega^2}{c^4 r}, \tag{5}$$

use Kepler's third law to write

$$a = \frac{(GM)^{1/3}}{\omega^{2/3}} \tag{6}$$

so that

$$h = \frac{2G^{5/3}\mu M^{2/3}\omega^{2/3}}{c^4 r}. \tag{7}$$

As a simple case, we can take a binary neutron star with $m_1 = m_2 = 1.4\,M_\odot$, and write $\omega = 2\pi/P$ where P is the orbital period. We find:

$$h = 2.8\times 10^{-21}\left(\frac{0.01\,\mathrm{s}}{P}\right)^{2/3}\frac{1\,\mathrm{Mpc}}{r}. \tag{8}$$

Since the rate of neutron star mergers in the Milky Way is very small (we have one supernova every ~ 50 years, and probably only a tiny fraction of these occur in binaries that leave two neutron stars in a tight orbit), in order to have a reasonable chance of detection in our lifetimes we must search many galaxies and thus many megaparsecs. Thus we must build detectors that are sensitive to strains of order 10^{-22}.

A. Measurement of the Hubble constant

Schutz (author of the text!) proposed an interesting method to measure the Hubble constant from merging neutron stars. Recall that gravitational wave detectors are really good at measuring the chirp mass, $\mathcal{M}_{\rm chirp} = \mu^{3/5}M^{2/5}$. Looking at Eq. (7), we see

$$h = \frac{2G^{5/3}\mathcal{M}_{\rm chirp}^{5/3}\omega^{2/3}}{c^4 r}. \tag{9}$$

Since h and ω are themselves GW observables, this means that if we observe a merging binary star, we can infer the distance r to the source. If the source can be localized to a particular galaxy (not trivial: GW detectors have poor angular resolution, as we will see – they are antennas, not telescopes), then with an optical spectrograph we can measure the redshift z. This is related to Hubble's constant H_0 by

$$cz = H_0 r, \tag{10}$$

and thus we can measure H_0 – historically a major challenge in cosmology, and a source of controversy even today.

The binary neutron star merger of August 14, 2017 made this method possible. The biggest challenge is that Eq. (7) is only valid for sources that are seen pole-on, and measuring the inclination angle of a gravitational wave source is challenging. For the 2017 event, electromagnetic observations of the jet produced during the merger helped to measure the inclination angle. In the future, networks of many gravitational wave detectors and hence precise measurements of both polarizations of gravitational wave will improve inclination measurements.

III. DETECTION OF GRAVITATIONAL WAVES

Let's now try to think about how gravitational waves might be detected; we will work in the TT gauge. A simple idea might be to consider two freely falling test masses with initially zero velocity relative to the coordinate system, $u^\alpha \to (1,0,0,0)$. Now in the TT gauge, the Christoffel symbols $\Gamma^\alpha{}_{00} = 0$, so u^α remains $(1,0,0,0)$ and the objects will stay at the same coordinate positions. If they are at a coordinate separation L (with $L \ll \lambda$) at angle ψ, then their spatial coordinate separation is

$$\Delta x^i \to (L\cos\psi, L\sin\psi, 0). \tag{11}$$

Now let's imagine that light travels from object A to object B; since $\Delta s^2 = 0$ for light, the coordinate time Δt for light to travel from one object to the other and back is

$$\begin{aligned} 0 = \Delta s^2 &= (\eta_{\mu\nu} + h_{\mu\nu})\Delta x^\mu \Delta x^\nu \\ &= -\Delta t^2 + (\delta_{ij} + h_{ij})\Delta x^i \Delta x^j \\ &= -\Delta t^2 + L^2 + 2h_+ L^2(\cos^2\psi - \sin^2\psi) + 4h_\times L^2 \cos\psi\sin\psi \\ &= -\Delta t^2 + L^2 + 2h_+ L^2\cos 2\psi + 2h_\times L^2 \sin 2\psi, \end{aligned} \tag{12}$$

so

$$\Delta t = \sqrt{L^2 + 2h_+ L^2\cos 2\psi + 2h_\times L^2\sin 2\psi} \approx L(1 + h_+\cos 2\psi + h_\times \sin 2\psi). \tag{13}$$

The time for light to go from A to B and back is twice this:

$$\Delta t_{\rm A\to B\to A} = 2L(1 + h_+\cos 2\psi + h_\times\sin 2\psi). \tag{14}$$

This is a coordinate time, but because $u^0 = 1$ for observer A, it is also the proper time seen by A. This light signal thus provides a means to detect a gravitational wave. The time delay due to the gravitational wave may be positive or negative. For the h_+ polarization, it has a maximum at $\psi = 0, 180°$ and a minimum at $\psi = 90, 270°$; for the $h_\times$ polarization, the pattern is rotated by 45°.

In a practical system such as LIGO, $L = 4$ km, and (putting in the obligatory factor of c) we find that the contribution of the gravitational wave to the time delay is $2Lh/c = 2.7\times10^{-27}$ s for $h = 10^{-22}$, which is a bit difficult in terms of timing. Instead, one can build an interferometer: we send light from A→B→A (separation at $\psi = 0$) and A→C→A (same L but now at $\psi = 90°$), and interfere the returning beams. This leads to a phase shift between the two paths:

$$\frac{2\pi}{\lambda}(\Delta t_{\rm A\to B\to A} - \Delta t_{\rm A\to C\to A}) = \frac{8\pi L h_+}{\lambda}, \tag{15}$$

which is 10^{-11} radians for $h_+ = 10^{-22}$ and $\lambda = 1.06\,\mu$m. The phase shift can be boosted further by reflecting the light many times on the A-B and A-C paths (hundreds; in practice done with a Fabry-Perot cavity). By operating the interferometer near a null (destructive interference fringe), and using a very high-powered laser beam injected at A, even small phase shifts can be measured. This is the principle behind LIGO and VIRGO. While the math seems simple enough, in practice these devices only work after an enormous and fascinating effort to chase down tiny sources of noise that are beyond the scope of this course.