# Lecture XIV: Gravitational waves: energy and evolution of binary systems

(Dated: October 18, 2019)

## I. OVERVIEW

Our next step is to consider the "energy" carried by gravitational waves. Part of the problem, of course, is defining exactly what that means, since by definition a vacuum solution to Einstein's equations has $T^{\mu\nu} = 0$. We will then consider the implications for evolution of binary stars. The most impressive consequence of gravitational waves is that – left on its own – a binary star cannot orbit forever, but will spiral in and merge as its orbital energy goes into gravitational waves.

## II. THE "ENERGY" OF A WAVE

A gravitational wave is in vacuum, and thus it has $T^{\mu\nu} = 0$: therefore an observer anywhere in the wave sees zero energy and momentum density, so there is no "energy" in the strictest sense of the word. Nevertheless, since Einstein's equations are not linear, if we consider a region of spacetime spanning many wavelengths of a gravitational wave, the presence of the wave can cause the large-scale geometry of the spacetime to curve. It is in this sense that we can say a wave has energy. Let us try to make this idea mathematically precise.

Suppose that we have a perturbed spacetime,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{1}$$

Then we may write Einstein's equations in vacuum as $G_{\mu\nu} = 0$. Now of course, we may expand $G_{\mu\nu}$ in powers of the perturbations $h_{\mu\nu}$:

$$G_{\mu\nu} = G^{(1)}{}_{\mu\nu} + G^{(2)}{}_{\mu\nu} + G^{(3)}{}_{\mu\nu} + \dots, \tag{2}$$

where $G^{(1)}{}_{\mu\nu}$ encapsulates all terms of order $h$, then $G^{(2)}{}_{\mu\nu}$ those of order $h^2$, etc. It does not take much imagination to see that to second order, one can write:

$$G^{(1)}{}_{\mu\nu} = -G^{(2)}{}_{\mu\nu} \tag{3}$$

and then define the effective stress-energy tensor

$$T^{\rm eff}{}_{\mu\nu} \equiv -\frac{1}{8\pi} G^{(2)}{}_{\mu\nu}. \tag{4}$$

In the case of interest here, we are interested in the effective stress-energy tensor that contributes to the large scale geometry of spacetime, incorporating the effect of the short-wavelength gravitational waves. In this case, what we want is really

$$T^{\rm eff}{}_{\mu\nu} \equiv -\frac{1}{8\pi} \langle G^{(2)}{}_{\mu\nu} \rangle, \tag{5}$$

where the average $\langle ... \rangle$ is intended to be taken over several wavelengths or more. It is only this effective stress-energy tensor that has a gauge-invariant meaning. We now set out to calculate it for gravitational waves.

### A. Calculation

Let's begin with the simplest case: a wave propagating in the $+z$ direction. In this case, $h_{\mu\nu}$ can be written as

$$h_{\mu\nu} \to \left( \begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 2h_+ & 2h_\times & 0 \\ 0 & 2h_\times & -2h_+ & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), \tag{6}$$

where we have used the TT gauge. This allows only two polarizations of gravitational wave, $+$ and $\times$. I will do the calculation for the $+$ polarization; the $\times$ polarization is obtained by rotating 45° and will give the same answer.

Propagation in the $z$-direction means that $h_+$ and $h_\times$ are functions of only $z-t$, so we can replace $\partial_3 \to -\partial_t$. All $\partial_1$ and $\partial_2$ derivatives vanish.

The metric is now diagonal, so

$$g_{00} = -1, \quad g_{11} = 1 + 2h_+, \quad g_{22} = 1 - 2h_+, \quad g_{33} = 1, \quad \text{others zero.} \tag{7}$$

The non-zero derivatives are:

$$g_{11,0} = 2\dot{h}_+, \quad g_{11,3} = -2\dot{h}_+, \quad g_{22,0} = -2\dot{h}_+, \quad g_{22,3} = 2\dot{h}_+, \quad \text{others zero.} \tag{8}$$

Then the Christoffel symbols:

$$\Gamma^0{}_{11} = \dot{h}_+, \quad \Gamma^1{}_{10} = \Gamma^1{}_{01} = \frac{\dot{h}_+}{1+2h_+}, \quad \Gamma^3{}_{11} = \dot{h}_+, \quad \Gamma^1{}_{13} = \Gamma^1{}_{31} = -\frac{\dot{h}_+}{1+2h_+},$$

$$\Gamma^0{}_{22} = -\dot{h}_+, \quad \Gamma^2{}_{20} = \Gamma^2{}_{02} = -\frac{\dot{h}_+}{1-2h_+}, \quad \Gamma^3{}_{22} = -\dot{h}_+, \quad \text{and} \quad \Gamma^2{}_{23} = \Gamma^2{}_{32} = \frac{\dot{h}_+}{1-2h_+}. \tag{9}$$

Now we recall that

$$R_{\theta\nu} = \Gamma^\alpha{}_{\theta\alpha,\nu} - \Gamma^\alpha{}_{\theta\nu,\alpha} + \Gamma^\alpha{}_{\delta\alpha}\Gamma^\delta{}_{\theta\nu} - \Gamma^\alpha{}_{\delta\nu}\Gamma^\delta{}_{\theta\alpha}. \tag{10}$$

This leads to the Ricci tensor components:

$$R_{00} = \frac{(1+2h_+)\ddot{h}_+ - 2\dot{h}_+^2}{(1+2h_+)^2} + \frac{-(1-2h_+)\ddot{h}_+ - 2\dot{h}_+^2}{(1-2h_+)^2} - \frac{\dot{h}_+^2}{(1+2h_+)^2} - \frac{\dot{h}_+^2}{(1-2h_+)^2}, \tag{11}$$

and by following the sign flips between the Christoffel symbols containing 0 and 3 it turns out that $R_{03} = R_{30} = -R_{00}$ and $R_{33} = R_{00}$ (the other components – $R_{01}$, $R_{02}$, $R_{12}$, $R_{13}$, $R_{23}$ – are zero). Since we only want the answer to second order, we simplify $R_{00}$ by Taylor expansion:

$$R_{00} \approx -4h_+\ddot{h}_+ - 6\dot{h}_+^2. \tag{12}$$

We now have

$$R = g^{\mu\nu}R_{\mu\nu} = -R_{00} + R_{33} = 0, \tag{13}$$

so the Einstein tensor components are the same as the Ricci tensor components in this case. Thus the second-order Einstein tensor component is:

$$G^{(2)}{}_{00} = R_{00} \approx -4h_+\ddot{h}_+ - 6\dot{h}_+^2. \tag{14}$$

The average over several wavelengths can be simplified by noting that $\langle h_+\ddot{h}_+\rangle = -\langle\dot{h}_+^2\rangle$ (try writing this for a sinusoid, or use integration by parts), thus

$$\langle G^{(2)}{}_{00}\rangle \approx -2\langle\dot{h}_+^2\rangle. \tag{15}$$

It then follows that the effective stress-energy tensor has an energy density

$$\rho^{\text{eff}} = T^{\text{eff}}{}_{00} = -\frac{1}{8\pi}(-2\langle\dot{h}_+^2\rangle) = \frac{1}{4\pi}\langle\dot{h}_+^2\rangle. \tag{16}$$

In general, one should include the other polarization as well and write

$$\rho^{\text{eff}} = \frac{1}{4\pi}\langle\dot{h}_+^2 + \dot{h}_\times^2\rangle = \frac{1}{32\pi}\langle\dot{h}_{ij}^{\text{TT}} h_{ij}^{\text{TT}}\rangle. \tag{17}$$

This is the usual formula for the energy of a gravitational wave. The momentum density, energy flux, and momentum flux are the same, because of the special result $G_{03} = -G_{00}$, $G_{33} = G_{00}$ for this specific problem.

## III. THE ENERGY RADIATED BY A SYSTEM

Let's return to the formula for the gravitational wave amplitude emitted by a source:

$$\mathbf{h}^{\mathrm{TT}} = \frac{2}{r}\mathrm{e}^{i\Omega(r-t)}\left(\mathbf{\Pi}_\perp\ddot{\mathbf{Q}}\mathbf{\Pi}_\perp - \frac{1}{2}\mathbf{\Pi}_\perp\mathrm{Tr}[\mathbf{\Pi}_\perp\ddot{\mathbf{Q}}]\right), \tag{18}$$

where we recall $\mathbf{\Pi}_\perp = \mathbb{I}_{3\times3} - \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}^{\mathrm{T}}$. Using Eq. (17), we can determine the gravitational wave luminosity by integrating the flux over the area:

$$L_{\mathrm{GW}} = \int_{S^2}\rho^{\mathrm{eff}}r^2\,\mathrm{d}^2\hat{\boldsymbol{n}}. \tag{19}$$

Recalling that only the real part of $\mathbf{h}^{\mathrm{TT}}$ is real, so that we need a factor of $\frac{1}{2}$, we have – in matrix notation –

$$\begin{aligned}
L_{\mathrm{GW}} &= \frac{1}{2}\int_{S^2}\frac{1}{32\pi}\mathrm{Tr}|(\dot{\mathbf{h}}^{\mathrm{TT}})^2|r^2\,\mathrm{d}^2\hat{\boldsymbol{n}} \\
&= \frac{1}{2}\int_{S^2}\frac{1}{32\pi}\mathrm{Tr}\left|\frac{2}{r}\left(\mathbf{\Pi}_\perp\dddot{\mathbf{Q}}\mathbf{\Pi}_\perp - \frac{1}{2}\mathbf{\Pi}_\perp\mathrm{Tr}[\mathbf{\Pi}_\perp\dddot{\mathbf{Q}}]\right)\right|^2 r^2\,\mathrm{d}^2\hat{\boldsymbol{n}} \\
&= \frac{1}{16\pi}\int_{S^2}\mathrm{Tr}\left|\mathbf{\Pi}_\perp\dddot{\mathbf{Q}}\mathbf{\Pi}_\perp - \frac{1}{2}\mathbf{\Pi}_\perp\mathrm{Tr}[\mathbf{\Pi}_\perp\dddot{\mathbf{Q}}]\right|^2\,\mathrm{d}^2\hat{\boldsymbol{n}} \\
&= \frac{1}{16\pi}\int_{S^2}\left[\mathrm{Tr}(\mathbf{\Pi}_\perp\dddot{\mathbf{Q}}\mathbf{\Pi}_\perp\dddot{\mathbf{Q}}^*) - \mathrm{Tr}(\mathbf{\Pi}_\perp\dddot{\mathbf{Q}})\mathrm{Tr}(\mathbf{\Pi}_\perp\dddot{\mathbf{Q}}^*) + \frac{1}{4}\mathrm{Tr}\mathbf{\Pi}_\perp\left|\mathrm{Tr}(\mathbf{\Pi}_\perp\dddot{\mathbf{Q}}^*)\right|^2\right]\mathrm{d}^2\hat{\boldsymbol{n}} \\
&= \frac{1}{16\pi}\int_{S^2}\left[\mathrm{Tr}(\mathbf{\Pi}_\perp\dddot{\mathbf{Q}}\mathbf{\Pi}_\perp\dddot{\mathbf{Q}}^*) - \frac{1}{2}\mathrm{Tr}(\mathbf{\Pi}_\perp\dddot{\mathbf{Q}})\mathrm{Tr}(\mathbf{\Pi}_\perp\dddot{\mathbf{Q}}^*)\right]\mathrm{d}^2\hat{\boldsymbol{n}},
\end{aligned} \tag{20}$$

where we used $\mathrm{Tr}\mathbf{\Pi}_\perp = 2$.

The integral can be simplified by writing the trace in component language and recalling the rules

$$\int_{S^2}\mathrm{d}^2\hat{\boldsymbol{n}} = 4\pi, \quad \int_{S^2}\hat{n}_i\hat{n}_j\,\mathrm{d}^2\hat{\boldsymbol{n}} = \frac{4\pi}{3}\delta_{ij}, \quad \text{and} \quad \int_{S^2}\hat{n}_i\hat{n}_j\hat{n}_k\hat{n}_l\,\mathrm{d}^2\hat{\boldsymbol{n}} = \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{21}$$

from which we determine

$$\int_{S^2}\Pi_{\perp\,ij}\Pi_{\perp\,kl}\,\mathrm{d}^2\hat{\boldsymbol{n}} = \int_{S^2}(\delta_{ij} - \hat{n}_i\hat{n}_j)(\delta_{kl} - \hat{n}_k\hat{n}_l)\,\mathrm{d}^2\hat{\boldsymbol{n}} = \frac{4\pi}{15}(6\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{22}$$

Then Eq. (20) becomes

$$\begin{aligned}
L_{\mathrm{GW}} &= \frac{1}{16\pi}\int_{S^2}\left[\Pi_{\perp\,ij}\Pi_{\perp\,kl}\ddot{Q}_{jk}\ddot{Q}^*_{li} - \frac{1}{2}\Pi_{\perp\,ij}\Pi_{\perp\,kl}\ddot{Q}_{ij}\ddot{Q}^*_{kl}\right]\mathrm{d}^2\hat{\boldsymbol{n}} \\
&= \frac{1}{60}(6\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\left(\dddot{Q}_{jk}\dddot{Q}^*_{li} - \frac{1}{2}\dddot{Q}_{ij}\dddot{Q}^*_{kl}\right) \\
&= \frac{1}{10}\left(\dddot{Q}_{ij}\dddot{Q}^*_{ij} - \frac{1}{3}\dddot{Q}_{ii}\dddot{Q}^*_{jj}\right).
\end{aligned} \tag{23}$$

This is usually written in terms of averages over a cycle:

$$L_{\mathrm{GW}} = \frac{1}{5}\left\langle\dddot{Q}_{ij}\dddot{Q}_{ij} - \frac{1}{3}\dddot{Q}_{ii}\dddot{Q}_{jj}\right\rangle. \tag{24}$$

This is the "quadrupole formula" for gravitational wave emission, and applies to any system with slowly moving masses in bound orbits. It is the gravitational analogue of the dipole formula in electromagnetic theory, $L_{\mathrm{EM}} = \frac{2}{3}\langle|\ddot{\mathbf{d}}|^2\rangle$.

Sometimes we put back the factors of $G$ and $c$:

$$L_{\mathrm{GW}} = \frac{G}{5c^5}\left\langle\dddot{Q}_{ij}\dddot{Q}_{ij} - \frac{1}{3}\dddot{Q}_{ii}\dddot{Q}_{jj}\right\rangle. \tag{25}$$

*Comment* – One will sometimes see Eq. (24) written in terms of the traceless part of the quadrupole moment: $\bar{\mathbf{Q}} = \mathbf{Q} - \frac{1}{3}\mathbb{I}_{3\times3}\mathrm{Tr}\mathbf{Q}$: $L_{\mathrm{GW}} = \frac{1}{5}\langle\dddot{\bar{Q}}_{ij}\dddot{\bar{Q}}_{ij}\rangle$. Some algebra shows that these are equivalent.

### A. Energy loss

I have proved in the above that gravitational waves carry energy in the sense that a region of spacetime containing gravitational waves has large-scale curvature. I haven't proven that the radiating system loses that energy, and I won't in this lecture. I *will* comment a bit on some of the considerations that go into the proof.

First, just as gravitational waves contribute an effective stress-energy tensor that curves spacetime at second order, so does the curvature associated with Newtonian gravity. There is a second-order energy density of order the Christoffel symbols squared – i.e., of order $|\nabla\phi|^2$. Careful accounting, and a lot of algebra, shows that this implies that the gravitational energy that you learned about in Newtonian physics really gravitates, and contributes to the gravitational mass of a system. (It contributes negatively: the mass of a binary star is less than the sum of the masses of the stars.)

Another consideration is that at first order in perturbation theory, we have ${T^{\mu\nu}}_{,\nu} = 0$, and hence an energy integral such as $\mathcal{E} = \int T^{00}\,\mathrm{d}^3\mathbf{x}$ is conserved: $\dot{\mathcal{E}} = -\int {T^{0i}}_{,i}\,\mathrm{d}^3\mathbf{x}$ and we can apply integration by parts. At second order in perturbation theory, we have instead

$$ {T^{\mu\nu}}_{;\nu} = {T^{\mu\nu}}_{,\nu} + {\Gamma^\mu}_{\alpha\nu}T^{\alpha\nu} + {\Gamma^\nu}_{\alpha\nu}T^{\mu\alpha} = 0. \tag{26} $$

Thus $\dot{\mathcal{E}}$ as defined above has contributions from the Christoffel symbols; a system can thus gain or lose energy from the curvature of spacetime. Thus even though energy and momentum are locally conserved, in a curved spacetime there is no global notion of what is energy and what is momentum. Only in the case of an isolated system in asymptotically flat spacetime – where there is a Lorentz frame that is well defined as one approaches $\infty$ – can one even speak of "total energy" of a system. We will define that total energy to be the gravitational mass measured by Kepler's third law, and we will carefully study how it relates to concepts like gravitational potential energy in a future lecture (when we study spherical stars).

## IV. EVOLUTION OF BINARY STAR ORBITS

Let's now consider the evolution of a binary star on a circular orbit in the $xy$-plane. The stars are taken to have masses $m_1$ and $m_2$ and separation $a$. They orbit each other with an angular frequency $\omega$:

$$ \omega = \sqrt{\frac{G(m_1+m_2)}{a^3}}. \tag{27} $$

The energy of the system – including the kinetic energy of the stars and their Newtonian gravitational potential energy – is

$$ \mathcal{E} = m_1 + m_2 - \frac{Gm_1m_2}{2a}. \tag{28} $$

(The $\frac{1}{2}$ comes from the fact that the kinetic energy cancels half of the potential energy.)

Now let's consider the energy lost to gravitational wave emission. The quadrupole moment is

$$ \mathbf{Q} = \mu a^2 \begin{pmatrix} \cos^2\omega t & \cos\omega t\sin\omega t & 0 \\ \cos\omega t\sin\omega t & \sin^2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\mu a^2 \begin{pmatrix} 1+\cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & 1-\cos 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{29} $$

where $\mu = m_1m_2/(m_1+m_2)$ is the reduced mass, and we have used the separation vector $(a\cos\omega t, a\sin\omega t, 0)$. We take the third derivative,

$$ \dddot{\mathbf{Q}} = 4\mu a^2\omega^3 \begin{pmatrix} \sin 2\omega t & -\cos 2\omega t & 0 \\ -\cos 2\omega t & -\sin 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{30} $$

Then:

$$ L_{\rm GW} = \frac{G}{5c^5}\left\langle \dddot{Q}_{ij}\dddot{Q}_{ij} - \frac{1}{3}\dddot{Q}_{ii}\dddot{Q}_{jj} \right\rangle = \frac{32G\mu^2a^4\omega^6}{5c^5}. \tag{31} $$

We set this equal to $-\dot{\mathcal{E}} = -Gm_1m_2\dot{a}/2a^2$, which leads to

$$\dot{a} = \frac{32G\mu^2a^4\omega^6}{5c^5(-Gm_1m_2/2a^2)} = -\frac{64\mu^2a^6\omega^6}{5m_1m_2c^5} = -\frac{64G^3\mu^2(m_1+m_2)^3}{5m_1m_2c^5a^3}. \tag{32}$$

We can see that the orbit shrinks. If we move the $a^3$ to the left-hand side and integrate, we can see that

$$\frac{1}{4}a^4 = \frac{1}{4}a_{\rm init}^4 - \frac{64G^3\mu^2(m_1+m_2)^3}{5m_1m_2c^5}t, \tag{33}$$

which means that the orbit shrinks to zero and the stars must merge in a gravitational wave inspiral time:

$$t_{\rm GW} = \frac{5m_1m_2c^5a_{\rm init}^4}{256G^3\mu^2(m_1+m_2)^3} = \frac{5c^5a_{\rm init}^4}{256G^3m_1m_2(m_1+m_2)}. \tag{34}$$

If we use $a_{\rm init} = G^{1/3}(m_1+m_2)^{1/3}\omega_{\rm init}^{-2/3}$, this can be re-expressed as

$$t_{\rm GW} = \frac{5c^5(m_1+m_2)^{1/3}}{256G^{5/3}m_1m_2\omega_{\rm init}^{8/3}}. \tag{35}$$

For the case of two neutron stars, $m_1 = m_2 = 1.4M_\odot$, this leads to a merger in 10 Gyr if $P_{\rm init} = 2\pi/\omega_{\rm init} = 15\,{\rm hr}$. There are several known binary neutron stars in our Galaxy with shorter periods than this, and they will merge in a time shorter than the age of the Galaxy. At $t_{\rm GW} = 1$ minute before the merger, one finds $P = 0.07$ s; at 1 second before the merger, $P = 0.015$ s. This is an appropriate range for gravitational wave detectors (next lecture).

The gravitational wave frequency is $2\omega$, i.e., twice the frequency of the orbit, if one considers the oscillatory part of $\mathbf{Q}$. A gravitational wave detector can measure this frequency very accurately. For the mergers seen with LIGO, one can also measure how this frequency varies with time (the "chirp": the frequency increases right before the merger), and so one gets a constraint on the combination of the masses $(m_1+m_2)^{1/3}/m_1m_2$. One normally defines the *chirp mass*

$$\mathcal{M}_{\rm chirp} \equiv \left[\frac{m_1m_2}{(m_1+m_2)^{1/3}}\right]^{3/5}, \tag{36}$$

which is the best-constrained property of a LIGO event. Measurement of the individual masses requires higher-order relativistic corrections, and these are not as well constrained.