Lecture XII: Gravitational lensing in the weak-field regime

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I. OVERVIEW

We now consider our first application of general relativity: the deflection of light by a massive object. This is one of the famous tests of general relativity, since GR gives a value that is twice that of Newtonian theory. It was first measured during the total solar eclipse of 1919, and has now been measured to very high precision by many techniques.

II. THE PROBLEM

Let's consider the metric in a Newtonian (non-relativistic, in terms of both speeds of motion of the matter and depths of the gravitational potential wells) situation. We previously identified the potential as the 00 component of the metric perturbation: $h_{00} = -2\phi$. Now we will want to consider the motion of relativistic particles (photons), so we will need all the metric components and all the Christoffel symbols.

We already know that the equations of linearized gravity in Lorenz gauge tell us that

$$\Box \bar{h}_{\theta\nu} = -16\pi T_{\theta\nu}, \quad h_{\theta\nu} = \bar{h}_{\theta\nu} - \frac{1}{2}\bar{h}\eta_{\mu\nu}. \tag{1}$$

In the Newtonian limit, only the 00 component of $\boldsymbol{T}$ is non-zero, so $\bar{h}_{00}$ is the only non-zero component of $\bar{\boldsymbol{h}}$. We conclude that $\bar{h} = \bar{h}^{\alpha}{}_{\alpha} = -\bar{h}_{00}$ and then

$$h_{00} = \bar{h}_{00} - \frac{1}{2}(-\bar{h}_{00})(-1) = \frac{1}{2}\bar{h}_{00}, \quad h_{0i} = 0, \quad h_{ij} = -\frac{1}{2}\bar{h}\delta_{ij} = \frac{1}{2}\bar{h}_{00}\delta_{ij}; \tag{2}$$

we then see that

$$h_{00} = -2\phi \quad \text{and} \quad h_{ij} = -2\phi\delta_{ij}. \tag{3}$$

Now this means that the metric is

$$\mathrm{d}s^2 = -(1+2\phi)\,\mathrm{d}t^2 + (1-2\phi)\,\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{4}$$

In the language of tests of general relativity, one often writes this as

$$\mathrm{d}s^2 = -(1+2\phi)\,\mathrm{d}t^2 + (1-2\gamma_{\mathrm{PPN}}\phi)\,\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \tag{5}$$

where γ_{PPN} is a *parameterized post-Newtonian* parameter. Its value is 1 in GR, would be 0 in a theory where gravity only distorts the time (00) part of the metric and not the spatial part, and could have other values in other theories. Measurements of gravitational lensing in the Solar System are usually quoted in terms of the measurement of the parameter γ_{PPN}. In what follows, I will assume GR for simplicity.

We now consider how two phenomena – gravitational lensing and the Shapiro time delay – work in this metric.

III. GRAVITATIONAL LENSING

A. Derivation

Let's consider a particle moving in the z-direction at velocity v, so its 4-momentum in the absence of the metric perturbation would be $\boldsymbol{u} \to (\gamma, 0, 0, v\gamma)$ with $\gamma = 1/\sqrt{1-v^2}$. (Obviously for lensing I care about the limit $v \to 1$.) We are interested in its deflection in the x-direction after passing near a massive object. To do this, we recall the geodesic equation

$$\frac{\mathrm{d}u^1}{\mathrm{d}\tau} = -\Gamma^1{}_{\alpha\beta}u^\alpha u^\beta. \tag{6}$$

We now use the *Born approximation*, valid to 1st order in h_{ij}, which states that since Γ is already a first-order perturbation we may plug in the unperturbed u^α:

$$\frac{\mathrm{d}u^1}{\mathrm{d}\tau} = -\Gamma^1{}_{00}\gamma^2 - 2\Gamma^1{}_{03}\gamma^2 v - \Gamma^1{}_{33}\gamma^2 v^2. \tag{7}$$

Now for the metric perturbation in Eq. (3), we have Christoffel symbols to linear order

$$\Gamma^1{}_{00} = \Gamma_{100} = -\frac{1}{2}h_{00,1} = \partial_1\phi, \quad \Gamma^1{}_{33} = \Gamma_{133} = -\frac{1}{2}h_{33,1} = \partial_1\phi, \quad \text{and} \quad \Gamma^1{}_{03} = \Gamma_{103} = 0. \tag{8}$$

This leads to the equation

$$\frac{\mathrm{d}u^1}{\mathrm{d}\tau} = -\gamma^2(1+v^2)\partial_1\phi. \tag{9}$$

We may divide by $\mathrm{d}u^3 = \mathrm{d}x^3/\mathrm{d}\tau$ to get the derivative with respect to the z-coordinate along the path:

$$\frac{\mathrm{d}u^1}{\mathrm{d}x^3} = \frac{\mathrm{d}u^1/\mathrm{d}\tau}{u^3} = \frac{-\gamma^2(1+v^2)\partial_1\phi}{\gamma v} = -\frac{\gamma(1+v^2)}{v}\partial_1\phi. \tag{10}$$

The change in u^1 along the path is

$$\Delta u^1 = -\frac{\gamma(1+v^2)}{v}\int \partial_1\phi(\boldsymbol{x})\,\mathrm{d}x^3, \tag{11}$$

and the change in angle in the 1-direction is this change divided by the pre-existing magnitude of the spatial part of $\boldsymbol{u}$ (that is, u^3):

$$\Delta\vartheta^1 = \frac{\Delta u^1}{u^3} = -\frac{1+v^2}{v^2}\int \partial_1\phi(\boldsymbol{x})\,\mathrm{d}x^3. \tag{12}$$

This is for a general potential. The most important case is where the potential is $\phi = -M/r$, where M is the mass of a point at the origin, and where a the particle passes by at $x^1 = b$ and $x^2 = 0$. The derivative of this is

$$\partial_1\phi(\boldsymbol{x}) = \partial_1\frac{-M}{\sqrt{(x^1)^2+(x^2)^2+(x^3)^2}} = \frac{M2x^1}{2[(x^1)^2+(x^2)^2+(x^3)^2]^{3/2}} = \frac{Mx^1}{[(x^1)^2+(x^2)^2+(x^3)^2]^{3/2}} = \frac{Mb}{[b^2+(x^3)^2]^{3/2}}. \tag{13}$$

Integrating over x^3 is possible with trigonometric substitution; the solution for Eq. (12) is

$$\Delta\vartheta^1 = -\frac{1+v^2}{v^2}\int_{-\infty}^{\infty}\frac{Mb}{[b^2+(x^3)^2]^{3/2}}\,\mathrm{d}x^3 = -\frac{1+v^2}{v^2}\frac{2M}{b} \to -\frac{4M}{b} \quad (v=1). \tag{14}$$

This is the formula for the deflection of light! Note that the Newtonian calculation would have been exactly the same, but without the $1+v^2$ in the numerator; thus for deflection of light, GR gives twice the Newtonian result.

B. Applications

When doing physics applications, it is easiest to put the factors of G and c back in Eq. (14):

$$|\Delta\vartheta| = \frac{4GM}{c^2 b}. \tag{15}$$

The first such application is the deflection of light by the Sun. For light that grazes the edge of the Sun, $M = 2\times10^{30}$ kg and $b = R_\odot = 7\times10^8$ m. This leads to an angle of

$$|\Delta\vartheta| = \frac{4(6.67\times10^{-11}\,\mathrm{m^3/kg\,s^2})(2\times10^{30}\,\mathrm{kg})}{(3\times10^8\,\mathrm{m/s})^2(7\times10^8\,\mathrm{m})} = 8.5\times10^{-6}\,\mathrm{rad} = 1.7\,\mathrm{arcsec}. \tag{16}$$

Of course, light that passes 2 solar radii from the center of the Sun gets half the deflection, and so on. Early twentieth century measurement techniques were sufficient to measure the positions of stars to better than an arcsec, but of course the glare of the Sun had to be blocked to do the measurement – hence the eclipse of 1919. Half a century later, radio interferometers observing quasars made a dramatic improvement in angular positioning, and they had the added advantage of not being blinded by sunlight. We are fortunate that the blazar 3C279, the brightest microwave point source in the sky, is only 12 arcminutes from the plane of Earth?s orbit and is eclipsed by the Sun once each year (on October 8). Measurements of the deflection of 3C279 using the continent-scale Very Long Baseline Array [Fomalont et al, ApJ 699:1395 (2009)] gives $\gamma_{\rm PPN} = 0.9998 \pm 0.0003$.

An even more extreme example of lensing is an *Einstein ring*. If a "lens" (mass M) is directly between the observer and the source, with distances $D_{\rm OL}$ from the observer to the lens and $D_{\rm LS}$ from the lens to the source, then the lens can focus rays from the source to a point if the deflection angle is (in the small-angle approximation)

$$|\Delta\vartheta| = \frac{b}{D_{\rm OL}} + \frac{b}{D_{\rm LS}} = b\left(\frac{1}{D_{\rm OL}} + \frac{1}{D_{\rm LS}}\right). \tag{17}$$

Comparing this to Eq. (15), we find

$$b = R_{\rm E} \equiv \sqrt{\frac{4GM}{c^2(1/D_{\rm OL} + 1/D_{\rm LS})}}. \tag{18}$$

This radius is called the *Einstein radius*. If the source is exactly behind the lens, then it really appears as a ring; if it is slightly misaligned, or the lens is not symmetrical (e.g., a galaxy or galaxy cluster), then one may instead see the source turned into multiple images or arcs.

Equation (18), of course, can only apply if the physical size of the lens is $< R_{\rm E}$. In the case of the Sun, if we place a source at ∞, then this requires $D_{\rm OL} > 8 \times 10^{15}\,{\rm m} = 550\,{\rm AU}$. So we can't see the Sun make an Einstein ring from Earth, or from anywhere out in space that our satellites have travelled. But out in the Galaxy, one star passing in front of another *can* make an Einstein ring as seen from the Solar System. The angular size of the ring is generally too small to see, but we can see that the source star becomes temporarily brighter – a phenomenon known as *microlensing*. The brightness vs. time can also be used to search for planets around the star. On even larger scales, whole galaxies or galaxy clusters can be lenses; in this case, one really does see the rings or arcs.

IV. THE SHAPIRO DELAY

Another effect that we may consider that is related to gravitational lensing is the *Shapiro delay* – the delay in arrival of a signal of the path from source to receiver passes near a massive object. This is closely related to lensing: in an ordinary lens, you know that the (phase) delay of light passing through a thick lens with index of refraction $n > 1$ is related to the fact that it causes the waves to converge. The same is true in a gravitational lens.

Let's consider again light traveling in the z-direction from $z = -\ell_1$ (source) to $z = \ell_2$ (receiver), with $\ell_1, \ell_2 \gg b$. Light travels on a null path, so

$$g_{\mu\nu}\frac{{\rm d}x^\mu}{{\rm d}t}\frac{{\rm d}x^\nu}{{\rm d}t} = 0; \tag{19}$$

for our metric, this means

$$-(1+2\phi) + (1-2\phi)\left[\left(\frac{{\rm d}x^1}{{\rm d}t}\right)^2 + \left(\frac{{\rm d}x^2}{{\rm d}t}\right)^2 + \left(\frac{{\rm d}x^3}{{\rm d}t}\right)^2\right] = 0. \tag{20}$$

Now ${\rm d}x^1/{\rm d}t$ and ${\rm d}x^2/{\rm d}t$ are both first order, so their squares are second order and can be dropped. Then:

$$-(1+2\phi) + (1-2\phi)\left(\frac{{\rm d}x^3}{{\rm d}t}\right)^2 = 0 \quad \rightarrow \quad \frac{{\rm d}x^3}{{\rm d}t} = \sqrt{\frac{1+2\phi}{1-2\phi}}. \tag{21}$$

It's more convenient here to talk about the travel time of a photon, so we write

$$\Delta t = \int_{-\ell_1}^{\ell_2} \frac{{\rm d}t}{{\rm d}x^3}\,{\rm d}x^3 = \int_{-\ell_1}^{\ell_2} \sqrt{\frac{1-2\phi}{1+2\phi}}\,{\rm d}x^3 = \int_{-\ell_1}^{\ell_2} (1-2\phi)\,{\rm d}x^3 = \ell_1 + \ell_2 - 2\int_{-\ell_1}^{\ell_2} \phi\,{\rm d}x^3. \tag{22}$$

The first two terms represent the expected travel time, based on the coordinate positions of the source and receiver. The third term is the Shapiro delay. As written, it seems a bit ambiguous; after all, the coordinate x^3 seems a bit arbitrary (it's a coordinate, not a physical object!), and we would probably measure the coordinate distance with a device that sends and receives signals (imaging "pinging" a space probe). To understand it better, let's re-write it in the way it is usually used.

We note that for impact parameter b,

$$\begin{aligned} \Delta t_{\rm Shapiro} = -2\int_{-\ell_1}^{\ell_2} \phi \, {\rm d}x^3 &= 2M\int_{-\ell_1}^{\ell_2} \frac{1}{\sqrt{b^2+(x^3)^2}}\,{\rm d}x^3 \\ &= 2M\left[\ln\left[\sqrt{b^2+(x^3)^2}+x^3\right]\right]_{-\ell_1}^{\ell_2} \\ &= 2M\ln\frac{\sqrt{b^2+\ell_2^2}+\ell_2}{\sqrt{b^2+\ell_1^2}-\ell_1} \\ &\approx 2M\ln\frac{2\ell_2}{b^2/(2\ell_1)} \\ &\approx -4M\ln\frac{b}{2\sqrt{\ell_1\ell_2}}. \end{aligned} \tag{23}$$

So the point is that if the source moves in orbit behind a massive object, then as the impact parameter b decreases to a minimum and comes back up (with ℓ_1 and ℓ_2 varying smoothly), the time delay reaches a maximum and goes back down. The absolute (constant) term is not independently measurable, but the coefficient of $\ln b$ is. With units put back in, we see

$$\Delta t_{\rm Shapiro} = {\rm constant} - \frac{4GM}{c^3}\ln b. \tag{24}$$

That coefficient for the mass of the Sun is 2×10^{-5} s – a very short amount of a time for a human, but easily measurable with radio ranging and atomic clocks. The Cassini space probe, flying almost behind the Sun as seen from Earth, has made an extremely accurate measurement of the Shapiro delay, using two frequencies to reduce systematic errors due to the index of refraction of the Sun's corona (the plasma delay, which preferentially affects lower frequencies). This gives rise to the most accurate determination of $\gamma_{\rm PPN}$ thus far: $\gamma_{\rm PPN} = 1.000021 \pm 0.000023$ [Bertotti, Iess & Tortora, Nature 425:374 (2003)].

The Shapiro delay can also be used in binary systems that contain a pulsar to measure the mass of the companion. The simplicity and robustness of the physics of the Shapiro delay (it does not depend on uncertain assumptions about magnetic fields or equations of state) is of particular importance. In the pulsar + white dwarf binary system PSR J1614−2230, with a nearly edge-on orbit, the Shapiro delay was used to measure the white dwarf mass and the inclination, and then (via Newton's third law) it was possible to obtain the pulsar mass as $1.97 \pm 0.04\,M_\odot$ [Demorest et al., Nature 467:1081 (2010)], thus definitively showing that neutron stars with such large masses are stable (we will explore the significance of that later in this course).