Lecture XI: The Einstein field equation

(Dated: October 2, 2019)

I. OVERVIEW

We are now ready to write down the Einstein field equations. This lecture corresponds approximately to Chapter 8 of the book (the material from Chapter 7 was distributed into the previous lectures).

II. MOTIVATION

We now want to write down a field equation, analogous to $F^{\mu\gamma}{}_{,\gamma} = J^\mu$ for Maxwell's equations in special relativity. Note that Maxwell's equations had the following properties:

1. The object that we used to describe electrodynamics was the 4-potential A_μ, a rank 1 tensor.

2. The 4-potential is not a directly measurable physical quantity because of a *gauge symmetry*, $A_\mu \to A_\mu + \partial_\mu \chi$ or $\boldsymbol{A} \to \boldsymbol{A} + \tilde{\boldsymbol{d}}\chi$ (where χ is any scalar field), which leaves the observable field $\boldsymbol{F} = \tilde{\boldsymbol{d}}\boldsymbol{A}$ (which can be measured by an observer with test charges) unchanged.

3. The equation of motion $A^{\gamma,\mu}{}_{,\gamma} - A^{\mu,\gamma}{}_{,\gamma} = J^\mu$ is a second-order PDE when written in terms of A_μ, linear in the second derivatives, and has the source J^μ on the right-hand side. [Comment: electromagnetism actually has a linear equation, but the equations of motion for the weak and strong forces are not linear; nevertheless, they are linear in the second derivatives.]

4. The source J^μ is locally conserved, $J^\mu{}_{,\mu} = 0$; this is not just a postulate, but is implied by the field equations ("automatic conservation of the source"). There is no consistent solution to Maxwell's equations in situations where charge is not conserved.

In a theory of curved spacetime, there are clear analogues to all of these items.

For property #1, the object we use to describe the geometry of spacetime will be the metric tensor $g_{\mu\nu}$, a rank 2 symmetric tensor.

For property #2, we know that there is also a gauge symmetry in general relativity: we can do a change of coordinates, and arrive at a different metric tensor (e.g., rectangular vs. polar coordinates in $\mathbb{R}^2$; all observable physics remains the same. The manifestation of geometry that is observable is the Riemann curvature tensor $R_{\hat\alpha\hat\beta\hat\gamma\hat\delta}$ in the frame of an observer at point $\mathcal{P}$ with their chosen orthonormal basis.

For property #3, we have a source $T^{\mu\nu}$, and we want to construct a tensor equation $M^{\mu\nu} = T^{\mu\nu}$. The object $\boldsymbol{M}$ must be a second-rank symmetric tensor, and we want it to be linear in the second derivatives of the metric (i.e., contain $g_{\alpha\beta,\gamma\delta}$ but not the square or other non-linear function thereof; it may be non-linear in $g_{\alpha\beta}$ or $g_{\alpha\beta,\gamma}$). This suggests that the 10 components of $\boldsymbol{M}$ be related to the 20 components of the Riemann tensor, somehow. We will see what this property implies for Einstein's equations. Note that relaxing this property (e.g., allowing 3rd or higher derivatives, allowing non-linear functions of the 2nd derivative, etc.) leads to more complicated theories known as *modified gravity* or *alternative gravity*.

For property #4, we know that if energy and momentum are locally conserved, we should have $T^{\mu\nu}{}_{;\nu} = 0$ in place of the special relativistic $T^{\mu\nu}{}_{,\nu} = 0$. (Note that there is no obvious notion here of global energy or momentum.) Therefore, whatever tensor $M^{\mu\nu}$ we construct, we want it to have $M^{\mu\nu}{}_{;\nu} = 0$ for any possible metric.

We now set out to see if we can construct such a tensor $\boldsymbol{M}$.

III. FINDING THE LEFT-HAND SIDE

We know that the 20 components of the second derivatives of the metric that are not just artifacts of the coordinate system are captured by the Riemann tensor $R_{\alpha\beta\gamma\delta}$. So how do we extract 10 components of this to become $M_{\mu\nu}$? We can make a tensor of lower rank by contraction against the metric or against the Levi-Civita tensor. However, we know from the symmetries of the Riemann tensor that $R_{\alpha[\beta\gamma\delta]} = 0$ so $\varepsilon^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = 0$; thus we can discard the latter idea. The contractions against the metric give

$$R_{\beta\delta} = g^{\alpha\gamma}R_{\alpha\beta\gamma\delta} = R^\alpha{}_{\beta\alpha\delta}; \tag{1}$$

there are actually $\binom{4}{2} = 6$ ways to choose 2 indices to contract but they all contain the same information by symmetry or are zero. The object $R_{\beta\delta}$ is known as the *Ricci tensor*; it is 2nd rank and symmetric. It has a further contraction, $R = g^{\beta\delta} R_{\beta\delta} = R^{\beta}{}_{\beta}$ known as the *Ricci scalar.*

If we want to construct a symmetric 2nd rank tensor from metric derivatives up through 2nd order and linear in the 2nd derivatives, then we have at our disposal the Ricci tensor, the Ricci scalar, and the metric tensor itself:

$$M^{\mu\nu} = c_1 R^{\mu\nu} + c_2 R g^{\mu\nu} + c_3 g^{\mu\nu}. \tag{2}$$

In order to get automatic conservation of the source, we need to study the divergence of $\boldsymbol{M}$. Since the divergence $g^{\mu\nu}{}_{;\nu} = 0$ (try writing this in a locally flat frame at any point $\mathcal{P}$), we have

$$M^{\mu\nu}{}_{;\nu} = c_1 R^{\mu\nu}{}_{;\nu} + c_2 R_{;\nu} g^{\mu\nu}. \tag{3}$$

To go further, let's use the Bianchi identity, $R_{\alpha\theta[\kappa\lambda;\tau]} = 0$, which is the main tool we have that says something useful about the covariant derivative of the Riemann tensor. Let's expand it:

$$R_{\alpha\theta\kappa\lambda;\tau} + R_{\alpha\theta\lambda\tau;\kappa} + R_{\alpha\theta\tau\kappa;\lambda} = 0, \tag{4}$$

and then multiply by $g^{\alpha\kappa}$:

$$g^{\alpha\kappa} R_{\alpha\theta\kappa\lambda;\tau} + g^{\alpha\kappa} R_{\alpha\theta\lambda\tau;\kappa} + g^{\alpha\kappa} R_{\alpha\theta\tau\kappa;\lambda} = R_{\theta\lambda;\tau} + R^{\alpha}{}_{\theta\lambda\tau;\alpha} - R_{\theta\tau;\lambda} = 0. \tag{5}$$

(In the last term, we used the antisymmetry of the Riemann tensor in the last two indices.) We multiply again by $g^{\theta\lambda}$:

$$g^{\theta\lambda} R_{\theta\lambda;\tau} + g^{\theta\lambda} R^{\alpha}{}_{\theta\lambda\tau;\alpha} - g^{\theta\lambda} R_{\theta\tau;\lambda} = R_{;\tau} - R^{\alpha}{}_{\tau;\alpha} - R^{\theta}{}_{\tau;\theta} = 0. \tag{6}$$

Rearranging gives

$$R^{\alpha}{}_{\tau;\alpha} = \frac{1}{2} R_{;\tau} \quad \text{or} \quad R^{\mu\nu}{}_{;\nu} = \frac{1}{2} R_{;\nu} g^{\mu\nu}. \tag{7}$$

This implies

$$M^{\mu\nu}{}_{;\nu} = \left(c_2 - \frac{1}{2} c_1 \right) g^{\mu\nu} R_{;\nu}. \tag{8}$$

Thus – unless the Ricci scalar is constant (which is true only for very special spacetimes), the automatic conservation of the source will hold if and only if $c_2 = \frac{1}{2} c_1$. We make this choice in what follows; then the Einstein field equation reads:

$$c_1 G^{\mu\nu} + c_3 g^{\mu\nu} = T^{\mu\nu}, \quad \text{where} \quad G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}. \tag{9}$$

We call $G^{\mu\nu}$ the *Einstein tensor.*

IV. THE COEFFICIENTS: NEWTON'S CONSTANT AND THE COSMOLOGICAL CONSTANT

We now wish to find the coefficients c_1 and c_3 in Einstein's equations. We first see that if $c_3 \neq 0$, then flat spacetime ($\boldsymbol{G} = 0$) cannot be empty ($\boldsymbol{T} = 0$). This is in conflict with our experience that there are vast, low-density regions of the Universe that are flat or have a very large radius of curvature, so we will set $c_3 = 0$ for the moment (certainly in the Solar System this is good). After deriving c_1, we will come back to the question of whether c_3 is really zero.

A. The equations of linearized gravity

Let's return to the problem of weakly perturbed spacetime, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. We agreed that to order h, the Christoffel symbols were

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} (-h_{\alpha\beta}{}^{,\mu} + h_{\alpha}{}^{\mu}{}_{,\beta} + h_{\beta}{}^{\mu}{}_{,\alpha}), \tag{10}$$

where the indices can be raised and lowered according to $\boldsymbol{\eta}$ since we work to order h. Then the Riemann tensor is

$$\begin{aligned} R^{\alpha}{}_{\theta\mu\nu} &= \Gamma^{\alpha}{}_{\theta\nu,\mu} - \Gamma^{\alpha}{}_{\theta\mu,\nu} + \mathcal{O}(\Gamma^2) \\ &= \frac{1}{2}(-h_{\theta\nu}{}^{,\alpha}{}_{,\mu} + h^{\alpha}{}_{\theta,\nu\mu} + h^{\alpha}{}_{\nu,\theta\mu} + h_{\theta\mu}{}^{,\alpha}{}_{,\nu} - h^{\alpha}{}_{\theta,\mu\nu} - h^{\alpha}{}_{\mu,\theta\nu}) \\ &= \frac{1}{2}(-h_{\theta\nu}{}^{,\alpha}{}_{,\mu} + h^{\alpha}{}_{\nu,\theta\mu} + h_{\theta\mu}{}^{,\alpha}{}_{,\nu} - h^{\alpha}{}_{\mu,\theta\nu}). \end{aligned} \tag{11}$$

(The Γ^2 terms are order h^2.) We can contract over α and μ to get the Ricci tensor:

$$R_{\theta\nu} = \frac{1}{2}(-\Box h_{\theta\nu} + h^{\alpha}{}_{\nu,\theta\alpha} + h_{\theta\alpha}{}^{,\alpha}{}_{,\nu} - h_{,\theta\nu}), \tag{12}$$

where $h \equiv h^{\alpha}{}_{\alpha}$ and

$$\Box \equiv \partial_\alpha \partial^\alpha = \nabla^2 - \partial_t^2. \tag{13}$$

A further contraction gives the Ricci scalar:

$$R = -\Box h + h^{\alpha\theta}{}_{,\alpha\theta}. \tag{14}$$

Then the Einstein tensor is

$$G_{\theta\nu} = R_{\theta\nu} - \frac{1}{2}Rg_{\theta\nu} = \frac{1}{2}(-\Box h_{\theta\nu} + h^{\alpha}{}_{\nu,\theta\alpha} + h_{\theta\alpha}{}^{,\alpha}{}_{,\nu} - h_{,\theta\nu} + \eta_{\theta\nu}\Box h - \eta_{\theta\nu} h^{\alpha\gamma}{}_{,\alpha\gamma}). \tag{15}$$

We thus arrive at the result:

$$-\Box h_{\theta\nu} + h^{\alpha}{}_{\nu,\theta\alpha} + h_{\theta\alpha}{}^{,\alpha}{}_{,\nu} - h_{,\theta\nu} + \eta_{\theta\nu}\Box h - \eta_{\theta\nu} h^{\alpha\gamma}{}_{,\alpha\gamma} = \frac{2}{c_1}T_{\theta\nu}. \tag{16}$$

This still looks like a somewhat messy equation. To go further, we choose a particular gauge. Recall that in electrodynamics, a gauge transformation was generated by a single function of spacetime (χ), and we could impose 1 gauge condition; a convenient choice for simplifying equations was the Lorenz gauge, $A^{\mu}{}_{,\mu} = 0$. In general relativity, since we have 4 coordinate functions x^μ of spacetime that we can adjust, we can impose 4 gauge conditions. We define the *trace-reversed perturbation*

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}h\eta_{\alpha\beta}, \quad \bar{h} \equiv \bar{h}^{\alpha}{}_{\alpha} = -\frac{D-2}{2}h, \quad h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \frac{1}{D-2}\bar{h}\eta_{\alpha\beta}. \tag{17}$$

(Recall that $\eta^{\alpha}{}_{\alpha} = \delta^{\alpha}{}_{\alpha} = D = 4$.) The *Lorenz gauge condition for weak-field gravity* is

$$\bar{h}^{\alpha\beta}{}_{,\beta} = 0. \tag{18}$$

(You can check this is 4 conditions for 4 coordinate degrees of freedom; I won't spend class time proving they are the "right" 4 conditions.) This implies $h^{\alpha\beta}{}_{,\beta} = \frac{1}{2}h^{,\alpha}$. Now Eq. (16) becomes

$$-\Box h_{\theta\nu} + \frac{1}{2}h_{,\nu\theta} + \frac{1}{2}h_{,\theta\nu} - h_{,\theta\nu} + \eta_{\theta\nu}\Box h - \frac{1}{2}\eta_{\theta\nu}\Box h = -\Box\bar{h}_{\theta\nu} = \frac{2}{c_1}T_{\theta\nu}. \tag{19}$$

Thus

$$\Box\bar{h}_{\theta\nu} = -\frac{2}{c_1}T_{\theta\nu}. \tag{20}$$

B. Newton's constant

In this gauge, there is a simple solution: a given component of $\boldsymbol{T}$ maps into the corresponding component of $\bar{\boldsymbol{h}}$. We will use Eq. (20) in full generality again later, but right now our objective is to understand the implications for non-relativistic systems such as the Solar System. Here we have energy density ρ dominating over other components of the stress-energy tensor, so the dominant component of $\bar{h}_{\theta\nu}$ is $\bar{h}_{00}$. It satisfies the equation

$$(\nabla^2 - \partial_t^2)\bar{h}_{00} = -\frac{2}{c_1}\rho \quad \to \bar{h}_{00} = -\frac{2}{c_1}\nabla^{-2}\rho. \tag{21}$$

We previously identified the gravitational potential as (specializing to $D = 4$ dimensions):

$$\phi = -\frac{1}{2}h_{00} = -\frac{1}{2}\left(\bar{h}_{00} - \frac{1}{2}\bar{h}\eta_{00}\right) = -\frac{1}{2}\left(\bar{h}_{00} - \frac{1}{2}\bar{h}_{00}\right) = -\frac{1}{4}\bar{h}_{00}. \tag{22}$$

Since $\phi = 4\pi\nabla^{-2}\rho$ in Newtonian gravity (with Newton's G set equal to 1), we must identify

$$4\pi\nabla^{-2}\rho = -\frac{1}{4}\bar{h}_{00} = -\frac{1}{4}\left(-\frac{2}{c_1}\nabla^{-2}\rho\right) \quad \to \quad c_1 = \frac{1}{8\pi}. \tag{23}$$

It is conventional to define $\Lambda = c_3/c_1$, so we write

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = 8\pi T^{\mu\nu}. \tag{24}$$

This is the *Einstein field equation* (EFE).

C. The cosmological constant

When general relativity was introduced, it would have been a natural assumption that special relativity was a solution to the EFE with no matter. After all, $\boldsymbol{F} = 0$ is a solution to Maxwell's equations with no charges. (It is not the only solution, since there is electromagnetic radiation; we will consider gravitational radiation soon.) In this case, we should have $\Lambda = 0$. But only experiment (or observation) can tell us whether this is really the case.

What experimental consequences might Λ have? To see this, let's return to linearized gravity, and insert Λ into Eq. (20):

$$\Box\bar{h}_{\theta\nu} - \frac{2c_3}{c_1}\eta_{\theta\nu} = -\frac{2}{c_1}T_{\theta\nu} \quad \to \quad \Box\bar{h}_{\theta\nu} = -16\pi T_{\theta\nu} + 2\Lambda\eta_{\theta\nu}. \tag{25}$$

If we have no matter, and demand a time-independent, spherically symmetric solution, this implies

$$\nabla^2\bar{h}_{\theta\nu} = 2\Lambda\eta_{\theta\nu}, \tag{26}$$

where the solution is now

$$\bar{h}_{\theta\nu} = \frac{1}{3}\Lambda r^2\eta_{\theta\nu}, \tag{27}$$

where $r^2 \equiv (x^1)^2 + (x^2)^2 + (x^3)^2$ is the squared radius. The trace reverse formula then implies

$$h_{00} = \bar{h}_{00} - \frac{1}{2}\bar{h}\eta_{00} = -\frac{1}{3}\Lambda r^2 - \frac{1}{2}\left(\frac{1}{3}\Lambda r^2 \times 4\right)(-1) = \frac{1}{3}\Lambda r^2, \tag{28}$$

so this term creates a gravitational potential $\phi = -\frac{1}{2}h_{00} = -\frac{1}{6}\Lambda r^2$. That is, if $\Lambda > 0$, then all particles accelerate away from the origin with an acceleration of $\frac{1}{3}\Lambda r$. Of course, all observers are freely falling and this does not imply anything special about the origin; everyone thinks everyone is accelerating away from everyone else. Two test particles originally near each other will then drift away exponentially ($\ddot{r} = \frac{1}{3}\Lambda r$, so $r \propto \exp\sqrt{\Lambda/3}\,t$). Since it is most important over long timescales, Λ is called the *cosmological constant*. The timescale $\sqrt{3/\Lambda}$ must be far longer than the orbital periods of the planets in order for the Solar System to be described by Newtonian gravity; it must even be longer than the orbital periods of the stars in the Milky Way in order for our galaxy to not fall apart. But Λ does appear to be positive according to cosmological observations that show the expansion of the Universe accelerating.

Over length scales $> \sqrt{3/\Lambda}$, it is clear that the linearized description of gravity must break down. Fortunately, with the fully non-linear Einstein equations, we will be able to solve for what really happens on such large scales in the Universe (at least under the assumption that Λ is the correct description of cosmic acceleration).