Lecture X: Curvature

(Dated: September 20, 2019)

I. OVERVIEW

This lecture, which corresponds to most of Chapter 6 of the book, concerns curvature of space or spacetime.

II. LOCALLY FLAT COORDINATES

We recall that on any smooth surface, at any point $\mathcal{P}$, it is possible to build an orthonormal set of basis vectors $\{\boldsymbol{e}_{\hat{\alpha}}\}$. It is then possible to build a barred coordinate system $x^{\bar{\alpha}}$ in which (i) $\mathcal{P}$ is at the origin, and (ii) $g_{\bar{\alpha}\bar{\beta}} = \eta_{\bar{\alpha}\bar{\beta}}$ (or $\delta_{\bar{\alpha}\bar{\beta}}$) at $\mathcal{P}$. The explicit construction was:

$$x^\mu = x^\mu(\mathcal{P}) + (\boldsymbol{e}_{\hat{\alpha}})^\mu x^{\bar{\alpha}}, \tag{1}$$

with

$$g_{\bar{\alpha}\bar{\beta}}(0) = \frac{\partial x^\mu}{\partial x^{\bar{\alpha}}}\frac{\partial x^\nu}{\partial x^{\bar{\beta}}} g_{\mu\nu} = (\boldsymbol{e}_{\hat{\alpha}})^\mu (\boldsymbol{e}_{\hat{\beta}})^\nu g_{\mu\nu} = \boldsymbol{e}_{\hat{\alpha}} \cdot \boldsymbol{e}_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}. \tag{2}$$

It is thus clear that numerical values of the metric tensor at a particular point are not any indication of how spacetime is curved.

We learned in the last lecture that the Christoffel symbols describe how the coordinate system is curved; they are related to the first derivative of the metric. Of course, these too depend on the coordinate system, since in Eq. (1) we could change the second derivatives of x^μ with respect to $x^{\bar{\alpha}}$. We may expand:

$$x^\mu = x^\mu(\mathcal{P}) + (\boldsymbol{e}_{\hat{\alpha}})^\mu x^{\bar{\alpha}} + \frac{1}{2}A^\mu{}_{\bar{\alpha}\bar{\beta}} x^{\bar{\alpha}} x^{\bar{\beta}} + \frac{1}{3!}B^\mu{}_{\bar{\alpha}\bar{\beta}\bar{\gamma}} x^{\bar{\alpha}} x^{\bar{\beta}} x^{\bar{\gamma}} + \dots; \tag{3}$$

The metric tensor in the barred frame is:

$$g_{\bar{\alpha}\bar{\beta}} = \frac{\partial x^\mu}{\partial x^{\bar{\alpha}}}\frac{\partial x^\nu}{\partial x^{\bar{\beta}}} g_{\mu\nu}, \quad \frac{\partial x^\mu}{\partial x^{\bar{\alpha}}} = (\boldsymbol{e}_{\hat{\alpha}})^\mu + A^\mu{}_{\bar{\alpha}\bar{\beta}} x^{\bar{\beta}} + \frac{1}{2}B^\mu{}_{\bar{\alpha}\bar{\beta}\bar{\gamma}} x^{\bar{\beta}} x^{\bar{\gamma}} + \dots. \tag{4}$$

One can see that if we Taylor expand the metric tensor around the origin in the barred coordinate system (i.e., around $\mathcal{P}$), then the $D^2(D+1)/2 = 40$ (for $D = 4$) first derivatives of the metric are related to the $D^2(D+1)/2 = 40$ components of $A^\mu{}_{\bar{\alpha}\bar{\beta}}$. Therefore, there are enough independent degrees of freedom in $A^\mu{}_{\bar{\alpha}\bar{\beta}}$ to make the metric derivatives, and hence the Christoffel symbols, zero at $\mathcal{P}$. One has to be careful of whether $A^\mu{}_{\bar{\alpha}\bar{\beta}}$ contains the "right" 40 components (40 equations and 40 unknowns don't guarantee a solution), but it turns out it does. Therefore, we conclude that at any point $\mathcal{P}$ we can choose a coordinate system where at that point $g_{\bar{\alpha}\bar{\beta}} = \eta_{\bar{\alpha}\bar{\beta}}$ and $g_{\bar{\alpha}\bar{\beta},\bar{\delta}} = 0$. **The Christoffel symbols describe the curvature of the coordinate system only, not the spacetime.**

Something else happens at the next order. The second derivatives of $g_{\bar{\alpha}\bar{\beta}}$ can change depending on how we choose the coefficients $B^\mu{}_{\bar{\alpha}\bar{\beta}\bar{\gamma}}$. There are $[D(D+1)/2]^2 = 100$ second derivatives of the metric, $g_{\bar{\alpha}\bar{\beta},\bar{\gamma}\bar{\delta}}$ (counting the fact that this is symmetric in both $\bar{\alpha}\bar{\beta}$ and $\bar{\gamma}\bar{\delta}$). However, there are $D^2(D+1)(D+2)/6 = 80$ coefficients $B^\mu{}_{\bar{\alpha}\bar{\beta}\bar{\gamma}}$ (counting the fact that there is symmetry in $\bar{\alpha}\bar{\beta}\bar{\gamma}$; in general if we have k symmetric indices, there are $(D+k-1)!/[D!(k-1)!]$ independent components, so there are $D(D+1)(D+2)/6$ independent third partial derivatives in the Taylor expansion of each x^μ). This means that of the second derivatives of the metric, there are

$$\left[\frac{D(D+1)}{2}\right]^2 - \frac{D^2(D+1)(D+2)}{6} = \frac{D^2(D^2-1)}{12} = \begin{cases} 1 & D=2 \\ 6 & D=3 \\ 20 & D=4 \end{cases} \tag{5}$$

components that can't be eliminated by changing the coordinate system. These 20 components represent the intrinsic curvature of spacetime. Thus, when we formalize the notion of "curvature," we want to work with the second derivatives of the metric, and we want a tensor with 20 independent components.

III. THE RIEMANN TENSOR

We will now approach the concept of curvature in a different way, without trying to build a special coordinate system. This will lead us to the Riemann tensor; we will define it first, using covariant derivatives of vectors, then look at its properties, and finally look at a few alternative constructions of the same object.

We want to build in a covariant way a second derivative of the metric, or a derivative of something containing a Christoffel symbol. The simplest candidate is to take the second covariant derivative of a vector,

$$\begin{aligned}\nabla_\mu \nabla_\nu v^\alpha &= \nabla_\mu(\partial_\nu v^\alpha + \Gamma^\alpha{}_{\theta\nu} v^\theta) \\ &= \partial_\mu(\partial_\nu v^\alpha + \Gamma^\alpha{}_{\theta\nu} v^\theta) - \Gamma^\beta{}_{\nu\mu}(\partial_\beta v^\alpha + \Gamma^\alpha{}_{\theta\beta} v^\theta) + \Gamma^\alpha{}_{\delta\mu}(\partial_\nu v^\delta + \Gamma^\delta{}_{\theta\nu} v^\theta) \\ &= \partial_\mu \partial_\nu v^\alpha + \Gamma^\alpha{}_{\theta\nu,\mu} v^\theta + \Gamma^\alpha{}_{\theta\nu} \partial_\mu v^\theta - \Gamma^\beta{}_{\nu\mu} \partial_\beta v^\alpha - \Gamma^\beta{}_{\nu\mu} \Gamma^\alpha{}_{\theta\beta} v^\theta + \Gamma^\alpha{}_{\delta\mu} \partial_\nu v^\delta + \Gamma^\alpha{}_{\delta\mu} \Gamma^\delta{}_{\theta\nu} v^\theta. \end{aligned} \tag{6}$$

This is a rank $\binom{1}{2}$ tensor, but unfortunately it is pretty messy. The interesting fact, though, is that unlike the second covariant derivative of a scalar, the second covariant derivative of a vector is not symmetric. In the final expression, the 1st, 4th, and 5th of the 7 terms are symmetric in μ and ν. The 3rd and 6th terms change into each other if we swap μ and ν. However, the 2nd and 7th terms are not symmetric. Instead:

$$\nabla_\mu \nabla_\nu v^\alpha - \nabla_\nu \nabla_\mu v^\alpha = (\Gamma^\alpha{}_{\theta\nu,\mu} - \Gamma^\alpha{}_{\theta\mu,\nu} + \Gamma^\alpha{}_{\delta\mu}\Gamma^\delta{}_{\theta\nu} - \Gamma^\alpha{}_{\delta\nu}\Gamma^\delta{}_{\theta\mu}) v^\theta \equiv R^\alpha{}_{\theta\mu\nu} v^\theta. \tag{7}$$

This is a covariant construction, yielding a tensor for any vector $\boldsymbol{v}$, so the object in parentheses must also be a tensor. It is called the *Riemann tensor*, $R^\alpha{}_{\theta\mu\nu}$, and it contains the metric tensor and its derivatives up through second order. The Riemann tensor is the simplest tensor that describes the curvature of spacetime. It has units of inverse length squared.

A. Symmetry properties

The Riemann tensor appears to have D^4 components (64 in $D = 4$). However there are some symmetry properties that reduce the number of independent components.

It is easiest to derive these properties by going to a locally flat coordinate system, where $g_{\alpha\beta,\gamma}(0) = 0$, and then transforming back to a general coordinate system at the end. In the locally flat system, we have $\Gamma^\lambda{}_{\rho\sigma}(0) = 0$ and

$$\Gamma^\lambda{}_{\rho\sigma,\tau}(0) = \partial_\tau \left\{ \frac{1}{2} g^{\lambda\kappa}(-g_{\rho\sigma,\kappa} + g_{\rho\kappa,\sigma} + g_{\sigma\kappa,\rho}) \right\}(0) = \frac{1}{2} g^{\lambda\kappa}(0)[-g_{\rho\sigma,\kappa\tau}(0) + g_{\rho\kappa,\sigma\tau}(0) + g_{\sigma\kappa,\rho\tau}(0)]. \tag{8}$$

(The derivatives of the inverse metric vanish.) It follows that, again at the origin, the $\Gamma\Gamma$ terms in the Riemann tensor go away and we have:

$$\begin{aligned} R^\alpha{}_{\theta\mu\nu}(0) &= \frac{1}{2} g^{\alpha\beta}(0)[-g_{\theta\nu,\beta\mu}(0) + g_{\theta\beta,\nu\mu}(0) + g_{\nu\beta,\theta\mu}(0)] - (\mu \leftrightarrow \nu) \\ &= \frac{1}{2} g^{\alpha\beta}(0)[-g_{\theta\nu,\beta\mu}(0) + g_{\nu\beta,\theta\mu}(0) + g_{\theta\mu,\beta\nu}(0) - g_{\mu\beta,\theta\nu}(0)], \end{aligned} \tag{9}$$

so by lowering an index:

$$R_{\beta\theta\mu\nu}(0) = \frac{1}{2}[-g_{\theta\nu,\beta\mu}(0) + g_{\nu\beta,\theta\mu}(0) + g_{\theta\mu,\beta\nu}(0) - g_{\mu\beta,\theta\nu}(0)]. \tag{10}$$

It follows by inspection of this result and the recollection that the metric tensor and second partial derivatives are symmetric that at the origin in the locally flat system:

- The Riemann tensor is antisymmetric in its last two indices: $R_{\beta\theta\mu\nu} = -R_{\beta\theta\nu\mu}$.
- The Riemann tensor is antisymmetric in its first two indices: $R_{\beta\theta\mu\nu} = -R_{\theta\beta\mu\nu}$.
- The Riemann tensor is symmetric under swapping the pairs of indices: $R_{\beta\theta\mu\nu} = R_{\mu\nu\beta\theta}$.
- The part of the Riemann tensor that is fully antisymmetric in its last three indices vanishes: $R_{\beta[\theta\mu\nu]} = 0$.

But since the Riemann tensor is a covariant construction, these symmetry properties are true everywhere in any coordinate system.

These symmetry properties constrain the Riemann tensor to have the requisite number of independent components (20 in 4 dimensions). Thus all of the second derivatives of the metric tensor that can't be eliminated by coordinate transformations are part of the Riemann tensor.

B. Bianchi identity

In flat-spacetime electromagnetism, we learned that the field equation $F^{\gamma\mu}{}_{,\gamma} = -J^\mu$ automatically implied the conservation of charge, $J^\mu{}_{,\mu} = 0$, because of the identity $F^{\gamma\mu}{}_{,\gamma\mu} = 0$ for any antisymmetric tensor $\mathbf{F}$. In general relativity, we will require the stress-energy tensor to be equal to certain components of the Riemann tensor. We want the field equations of general relativity to imply the conservation of 4-momentum in the same way, $T^{\mu\nu}{}_{;\nu} = 0$ (here with a covariant instead of partial derivative). This means we need an identity that sets some combination of derivatives of the Riemann tensor to zero for any spacetime.

The identity that we need for this construction is the (uncontracted) Bianchi identity. If we go to a locally flat frame, where $\Gamma = 0$, then the derivative of the Riemann tensor is

$$\begin{aligned} R^\alpha{}_{\theta\mu\nu;\tau}(0) &= R^\alpha{}_{\theta\mu\nu,\tau}(0) \\ &= \Gamma^\alpha_{\theta\nu,\mu\tau}(0) - (\mu \leftrightarrow \nu) \\ &= \frac{1}{2}[g^{\alpha\beta}(-g_{\theta\nu,\beta} + g_{\beta\nu,\theta} + g_{\beta\theta,\nu})]_{,\mu\tau}(0) - (\mu \leftrightarrow \nu) \\ &= \frac{1}{2}g^{\alpha\beta}(0)[-g_{\theta\nu,\beta\mu\tau} + g_{\beta\nu,\theta\mu\tau} + g_{\beta\theta,\nu\mu\tau}](0) - (\mu \leftrightarrow \nu) \\ &= \frac{1}{2}g^{\alpha\beta}(0)[-g_{\theta\nu,\beta\mu\tau} + g_{\theta\mu,\beta\nu\tau} + g_{\beta\nu,\theta\mu\tau} - g_{\beta\mu,\theta\nu\tau}](0). \end{aligned} \tag{11}$$

Now if we antisymmetrize in the last three indices, and use the symmetry of partial derivatives, we get

$$R^\alpha{}_{\theta[\mu\nu;\tau]} = 0. \tag{12}$$

This is a covariant object, so if it vanishes in one coordinate system it vanishes in all. Since the Riemann tensor is already antisymmetric in μ and ν, we can expand the antisymmetrization as:

$$R^\alpha{}_{\theta\mu\nu;\tau} + R^\alpha{}_{\theta\tau\mu;\nu} + R^\alpha{}_{\theta\nu\tau;\mu} = 0. \tag{13}$$

This is the *Bianchi identity.*

IV. APPLICATION TO POLAR COORDINATES

Before we continue, let's go back to polar coordinates. The Riemann tensor should be zero, since polar coordinates are actually a description of flat $\mathbb{R}^2$. In $D = 2$ dimensions, the antisymmetrization rules mean that there is only one nontrivial component of the Riemann tensor; we will thus compute $R^r{}_{\phi r\phi}$.

We recall the non-zero Christoffel symbols:

$$\Gamma^r{}_{\phi\phi} = -r, \quad \Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{1}{r}, \quad \text{others zero.} \tag{14}$$

Then:

$$\begin{aligned} R^r{}_{\phi r\phi} &= \Gamma^r{}_{\phi\phi,r} - \Gamma^r{}_{\phi r,\phi} + \Gamma^r{}_{\delta r}\Gamma^\delta{}_{\phi\phi} - \Gamma^r{}_{\delta\phi}\Gamma^\delta{}_{\phi r} \\ &= (-r)_{,r} - 0 + [0+0] - [0 + (-r)\frac{1}{r}] \\ &= -1 + 1 = 0. \end{aligned} \tag{15}$$

So indeed the Riemann tensor in polar coordinates is zero. This fact – that polar coordinates describe flat space – was derived entirely from the metric or line element, $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2$.

[Note: It is true – but I haven't actually proven it – that if the Riemann tensor is zero in some region, then that region is flat with the possible exception of topological identifications such as cylinders.]

V. THE EQUATION OF GEODESIC DEVIATION

We now come to an alternative description of the Riemann tensor – the way it describes how neighboring geodesics deviate from each other. To do this, let's consider not a single geodesic, but a family of geodesics, $x^\mu(\lambda, \psi)$, where

λ is the parameter of the geodesic and ψ tells us which geodesic we are considering. We define the infinitesimal displacement between neighboring geodesics:

$$\xi^\mu(\lambda,\psi) = \frac{\mathrm{d}x^\mu(\lambda,\psi)}{\mathrm{d}\psi} = \lim_{\epsilon\to 0}\frac{x^\mu(\lambda,\psi+\epsilon)-x^\mu(\lambda,\psi)}{\epsilon}. \tag{16}$$

We similarly recall the 4-velocity $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$. Our objective is to determine how this infinitesimal displacement $\boldsymbol{\xi}$ changes as we move along a geodesic. Since the geodesic equation is itself second order, we expect to get a second-order equation, i.e., we expect to get an expression for

$$\mathrm{D}^2\boldsymbol{\xi}/\mathrm{d}\lambda^2 = \nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{u}}\boldsymbol{\xi}. \tag{17}$$

We first begin with a simple lemma on the vector fields: we see that $\nabla_{\boldsymbol{u}}\boldsymbol{\xi}$ has components:

$$\nabla_{\boldsymbol{u}}\xi^\alpha = u^\beta\xi^\alpha{}_{;\beta} = u^\beta\xi^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\gamma\beta}u^\beta\xi^\gamma = \frac{\partial\xi^\alpha}{\partial\lambda} + \Gamma^\alpha{}_{\gamma\beta}u^\beta\xi^\gamma = \frac{\partial}{\partial\lambda}\frac{\partial x^\alpha}{\partial\psi} + \Gamma^\alpha{}_{\gamma\beta}u^\beta\xi^\gamma. \tag{18}$$

This is the same as the α component of $\nabla_{\boldsymbol{\xi}}\boldsymbol{u}$, since switching $\boldsymbol{\xi}$ and $\boldsymbol{u}$ is equivalent to switching ψ and λ, and the partial derivatives commute. Therefore:

$$\nabla_{\boldsymbol{u}}\boldsymbol{\xi} = \nabla_{\boldsymbol{\xi}}\boldsymbol{u} \quad\text{or}\quad u^\beta\nabla_\beta\xi^\alpha = \xi^\beta\nabla_\beta u^\alpha. \tag{19}$$

We then write:

$$\begin{aligned}
\frac{\mathrm{D}^2\xi^\alpha}{\mathrm{d}\lambda^2} = \nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{u}}\xi^\alpha &= u^\gamma\nabla_\gamma(u^\beta\nabla_\beta\xi^\alpha) \\
&= u^\gamma\nabla_\gamma(\xi^\beta\nabla_\beta u^\alpha) \\
&= (u^\gamma\nabla_\gamma\xi^\beta)\nabla_\beta u^\alpha + \xi^\beta u^\gamma\nabla_\gamma\nabla_\beta u^\alpha \\
&= (\xi^\gamma\nabla_\gamma u^\beta)\nabla_\beta u^\alpha + \xi^\beta u^\gamma\nabla_\beta\nabla_\gamma u^\alpha + \xi^\beta u^\gamma R^\alpha{}_{\delta\gamma\beta}u^\delta \\
&= (\xi^\gamma\nabla_\gamma u^\beta)\nabla_\beta u^\alpha + \xi^\gamma u^\beta\nabla_\gamma\nabla_\beta u^\alpha + \xi^\beta u^\gamma R^\alpha{}_{\delta\gamma\beta}u^\delta \\
&= \xi^\gamma\nabla_\gamma(u^\beta\nabla_\beta u^\alpha) + \xi^\beta u^\gamma R^\alpha{}_{\delta\gamma\beta}u^\delta.
\end{aligned} \tag{20}$$

Now the first term is zero because of the geodesic equation. This means

$$\frac{\mathrm{D}^2\xi^\alpha}{\mathrm{d}\lambda^2} = R^\alpha{}_{\delta\gamma\beta}\xi^\beta u^\gamma u^\delta. \tag{21}$$

This is the *equation of geodesic deviation.*

A. Gravity gradients

The above calculation can be compared to what we normally experience in Newtonian gravity. In a locally flat coordinate system centered at point $\mathcal{P}$, with $\boldsymbol{u}$ in the time-direction $u^\alpha \to (1,0,0,0)$, the equation of geodesic deviation says

$$\frac{\mathrm{D}^2\xi^{\hat{i}}}{\mathrm{d}\tau^2} = R^{\hat{i}}{}_{\hat{0}\hat{0}\hat{j}}\xi^{\hat{j}}. \tag{22}$$

Thus $R^{\hat{i}}{}_{\hat{0}\hat{0}\hat{j}}$ is what in Newtonian physics we would call the gravitational field gradient $\partial g_i/\partial x^j$ $-\partial_i\partial_j\phi$. The symmetry properties of Riemann tell us that $R^{\hat{i}}{}_{\hat{0}\hat{0}\hat{j}}$ is symmetric under interchange of $\hat{i}$ and $\hat{j}$.

Of course this *gravity gradient* in the frame of an observer with 4-velocity $\boldsymbol{u}$ contains only 6 independent components; the other 14 of the components of the Riemann tensor tell us what the gravity gradient would be for a moving observer. If you want to use your electromagnetic intuition here, recall that $F^{\mu\nu}$ had 6 components, of which 3 (the electric field) described forces on stationary charged particles and 3 (the magnetic field) only applied to charges that were moving. The same thing happens here, except the gravity gradient is a more complicated object than the electric field (it has 6 instead of 3 components) and the part that applies to moving particles has 14 instead of 3 components.

In Newtonian gravity, we learned that gravity gradient had two types of components that behaved in a fundamentally different way. One component – the trace, $-\nabla^2\phi$ – was related to the local matter density by $-\nabla^2\phi = -4\pi\rho$. (Starting here, I will set Newton's gravitational constant $G_{\rm N} = 1$.) The other 5 components were the tidal field,

$$s_{ij} = -\partial_i\partial_j\phi + \frac{1}{3}\delta_{ij}\nabla^2\phi, \tag{23}$$

which formed a traceless-symmetric tensor. The tidal field was determined non-locally by the matter distribution, and can exist even in regions with no matter: two particles placed near each other could get pulled apart or squeezed together by gravity, even when the source of that gravity is somewhere else. We will see that something similar happens in GR.