Lecture IX: Geodesics and parallel transport

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I. OVERVIEW

We will now consider the computation of geodesics in a curved spacetime. We will do this both from the perspective of parallel transport (using derivatives) and from the variational principle.

II. DIRECTIONAL DERIVATIVES AND PARALLEL TRANSPORT

Let's consider a tensor $\mathbf{T}$, which can be measured at various points along a path $x^\mu(\sigma)$. This path has a tangent vector $\boldsymbol{u}$ with $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\sigma$. The directional derivative of $\mathbf{T}$ along the path is important enough to get a special symbol:

$$\frac{\mathrm{D}\mathbf{T}}{\mathrm{d}\sigma} \equiv \boldsymbol{\nabla}_{\boldsymbol{u}}\mathbf{T}. \tag{1}$$

Note that the directional derivative for a tensor could be written as

$$\frac{\mathrm{D}T^\alpha{}_\beta}{\mathrm{d}\sigma} = u^\gamma T^\alpha{}_{\beta;\gamma} = u^\gamma T^\alpha{}_{\beta,\gamma} + u^\gamma \Gamma^\alpha{}_{\delta\gamma} T^\delta{}_\beta - u^\gamma \Gamma^\delta{}_{\beta\gamma} T^\alpha{}_\delta = \frac{\mathrm{d}T^\alpha{}_{\beta,\gamma}}{\mathrm{d}\sigma} + u^\gamma \Gamma^\alpha{}_{\delta\gamma} T^\delta{}_\beta - u^\gamma \Gamma^\delta{}_{\beta\gamma} T^\alpha{}_\delta. \tag{2}$$

We use the capital D to denote that we take the directional derivative of the tensor and then this has components, whereas the lower-case d indicates that we take the components first and then take their derivatives.

If the directional derivative of $\mathbf{T}$ along the trajectory is zero, we say that $\mathbf{T}$ is *parallel transported*. The parallel transport of a tensor or vector along a path is of fundamental importance in describing curvature. It is the closest we can come to saying that a tensor or vector is "constant" along a trajectory.

III. GEODESICS DEFINED BY PARALLEL TRANSPORT

Now let's consider the "straightest possible path" in a spacetime: this is the path where the tangent vector $\boldsymbol{u}$ is itself parallel transported. A curve defined in this way is a *geodesic*. It obeys the relation:

$$0 = \frac{\mathrm{D}u^\alpha}{\mathrm{d}\sigma} = \frac{\mathrm{d}u^\alpha}{\mathrm{d}\lambda} + \Gamma^\alpha{}_{\beta\gamma} u^\beta u^\gamma. \tag{3}$$

Since $u^\alpha = \mathrm{d}x^\alpha/\mathrm{d}\lambda$ (I will call the parameter λ here instead of σ), we see that

$$0 = \frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} + \Gamma^\alpha{}_{\beta\gamma} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\gamma}{\mathrm{d}\lambda} \tag{4}$$

or

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} = -\Gamma^\alpha{}_{\beta\gamma} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\gamma}{\mathrm{d}\lambda}. \tag{5}$$

Since the Γ coefficients are themselves functions of the coordinates, this can be thought of as a 2nd order differential equation for the path of a particle following a geodesic. Note that $\boldsymbol{u}\cdot\boldsymbol{u}$ is conserved along a geodesic, since the directional derivative obeys the product rule:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\boldsymbol{u}\cdot\boldsymbol{u}) = \boldsymbol{u}\cdot\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{d}\lambda} + \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{d}\lambda}\cdot\boldsymbol{u} = 0. \tag{6}$$

The choice of normalization of λ is related to the normalization of $\boldsymbol{u}\cdot\boldsymbol{u}$. We will ordinarily set $\boldsymbol{u}\cdot\boldsymbol{u}$ to be $+1$ when we follow trajectories in Euclidean-signature metrics (e.g., on a curved surface embedded in 3-space, so that λ is the path length); -1 for following massive particles in spacetime (so that λ is the proper time); and 0 for following photon trajectories in spacetime (there is no other choice: their trajectories have zero length).

We now consider several examples of geodesics. In what follows, I will use the dot ($\dot{}$) to denote $\mathrm{d}/\mathrm{d}\lambda$ as a shorthand.

A. Example: polar coordinates

Recall that in polar coordinates, the non-zero Christoffel symbols are

$$\Gamma^r{}_{\phi\phi} = -r, \quad \Gamma^\phi{}_{\phi r} = \Gamma^\phi{}_{r\phi} = \frac{1}{r}. \tag{7}$$

This leads to the system of equations

$$\ddot{r} = r\dot{\phi}^2 \quad \text{and} \quad \ddot{\phi} = -\frac{2}{r}\dot{r}\dot{\phi}. \tag{8}$$

(Note the 2 since in the ϕ equation there is both a $\beta = r$, $\gamma = \phi$ term and a $\beta = \phi$, $\gamma = r$ term.) The radius equation contains a "centrifugal force" term: in the polar coordinate system, a geodesic appears to bend toward positive r, but this is an artifact of the coordinate system bending. Similarly, if $\dot{r} > 0$ and $\dot{\phi} > 0$, then $\ddot{\phi} < 0$: the motion in ϕ appears to slow down, but this is really due to the lines of constant ϕ getting spaced farther apart and becoming more parallel to the particle trajectory.

The conservation of $\boldsymbol{u} \cdot \boldsymbol{u}$ can be seen explicitly in this case:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\dot{r}^2 + r^2\dot{\phi}^2) = 2\dot{r}\ddot{r} + 2r\dot{r}\dot{\phi}^2 + 2r^2\dot{\phi}\ddot{\phi} = 2\dot{r}(r\dot{\phi}^2) + 2r\dot{r}\dot{\phi}^2 + 2r^2\dot{\phi}\left(-\frac{2}{r}\dot{r}\dot{\phi}\right) = 0; \tag{9}$$

following convention we will normalize to $\dot{r}^2 + r^2\dot{\phi}^2 = 1$. In this case, Eq. (8) admits a second conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(r^2\dot{\phi}) = r^2\ddot{\phi} + 2r\dot{r}\dot{\phi} = r^2\left(-\frac{2}{r}\dot{r}\dot{\phi}\right) + 2r\dot{r}\dot{\phi} = 0. \tag{10}$$

We will call this $\tilde{L} = r^2\dot{\phi}$.

The existence of the second conservation law enables us to solve for the geodesic in terms of analytic functions. We can see that

$$\dot{\phi} = \frac{\tilde{L}}{r^2} \quad \text{and} \quad \dot{r} = \pm\sqrt{1 - r^2\dot{\phi}^2} = \pm\sqrt{1 - \frac{\tilde{L}^2}{r^2}}. \tag{11}$$

We thus find the solution for relating r to λ is then

$$\lambda = \int \frac{\mathrm{d}r}{\dot{r}} = \int \pm\frac{\mathrm{d}r}{\sqrt{1 - \frac{\tilde{L}^2}{r^2}}} = \int \pm\frac{r\,\mathrm{d}r}{\sqrt{r^2 - \tilde{L}^2}} = \pm\sqrt{r^2 - \tilde{L}^2} + C. \tag{12}$$

Note only the region $r \geq |\tilde{L}|$ is allowed, and there is a "turnaround point" where $r = r_{\rm min} = |\tilde{L}|$ and $\lambda = C$: the geodesic trajectory comes in, reaches this minimum r, and then goes back out. Then we can also solve for ϕ:

$$\phi = \int \frac{\tilde{L}}{r^2}\,\mathrm{d}\lambda = \int \frac{\tilde{L}}{\tilde{L}^2 + (\lambda - C)^2}\,\mathrm{d}\lambda = \tan^{-1}\frac{\lambda - C}{\tilde{L}} + \phi_0, \tag{13}$$

where ϕ_0 is a constant of integration. It represents the longitude at the turnaround point. Of course, ϕ ranges from $\phi_0 - \frac{\pi}{2}$ to $\phi_0 + \frac{\pi}{2}$ (or the other way, if $\tilde{L} < 0$). This should be familiar from your description of straight lines in polar coordinates.

B. A note on symmetries

In the example above, you saw that there was a symmetry of the space: $g_{\alpha\beta}$ did not depend on ϕ. In general, this implies this implies a conservation law for p_ϕ for a particle, consistent with Noether's theorem. In this case, since we considered simply a geodesic trajectory, we expect u_ϕ to be conserved. This is true because:

$$\begin{aligned}
\dot{u}_\phi &= \frac{\mathrm{d}}{\mathrm{d}\lambda}(g_{\phi\alpha}u^\alpha) \\
&= \frac{\mathrm{d}g_{\phi\alpha}}{\mathrm{d}\lambda}u^\alpha + g_{\phi\alpha}\frac{\mathrm{d}u^\alpha}{\mathrm{d}\lambda} \\
&= g_{\phi\alpha,\beta}u^\beta u^\alpha - g_{\phi\alpha}\Gamma^\alpha{}_{\beta\gamma}u^\beta u^\gamma \\
&= g_{\phi\alpha,\beta}u^\beta u^\alpha - \Gamma_{\phi\beta\gamma}u^\beta u^\gamma \\
&= (g_{\phi\gamma,\beta} - \Gamma_{\phi\beta\gamma})u^\beta u^\gamma \\
&= \left(g_{\phi\gamma,\beta} - \frac{1}{2}g_{\phi\beta,\gamma} - \frac{1}{2}g_{\phi\gamma,\beta} + \frac{1}{2}g_{\beta\gamma,\phi}\right)u^\beta u^\gamma \\
&= \frac{1}{2}g_{\beta\gamma,\phi}u^\beta u^\gamma = 0,
\end{aligned} \tag{14}$$

where we used that $g_{\beta\gamma,\phi} = 0$ (metric components independent of ϕ) in the very last line. (The rest of Eq. 14 is true in general.)

The polar coordinate case thus has $u_\phi = r^2 u^\phi = r^2\dot{\phi}$ conserved.

Some special spacetimes have enough symmetries and hence conserved quantities that we can express the motion in terms of integrals (like in the above case). This will include most of the spacetimes we study, because we will focus on highly symmetrical cases where we can solve Einstein's equations exactly, but this is not possible for most of the spacetimes that can exist.

C. Weak-field gravity with slow-moving particles: the Newtonian limit

A case of direct relevance to GR consists of particles in a spacetime whose metric is approximately, but not exactly, the metric of special relativity:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \tag{15}$$

where $|h_{\alpha\beta}| \ll 1$. In this case, to first order in h:

$$\Gamma_{\mu\alpha\beta} = \frac{1}{2}(-h_{\alpha\beta,\mu} + h_{\alpha\mu,\beta} + h_{\beta\mu,\alpha}). \tag{16}$$

Now the inverse metric is $\eta^{\mu\nu}$ plus order-h corrections, so

$$\Gamma^0{}_{\alpha\beta} \approx \frac{1}{2}(h_{\alpha\beta,0} - h_{\alpha 0,\beta} - h_{\beta 0,\alpha}) \quad \text{and} \quad \Gamma^i{}_{\alpha\beta} \approx \frac{1}{2}(-h_{\alpha\beta,i} + h_{\alpha i,\beta} + h_{\beta i,\alpha}) \tag{17}$$

with corrections of order h^2.

Now if we have a slow-moving particle, with velocity $\ll 1$ so that u^α has components $\approx (1,0,0,0)$, then we see that

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}\tau^2} = -\Gamma^i{}_{\alpha\beta}u^\alpha u^\beta \approx -\Gamma^i{}_{00} \approx -\frac{1}{2}(-h_{00,i} + h_{0i,0} + h_{0i,0}) = \frac{1}{2}h_{00,i} - h_{0i,0}. \tag{18}$$

If the sources of gravity are also slow-moving, so that the time derivatives (which have an order of magnitude of $\sim h/$dynamical time) are small compared to the spatial derivatives (which have an order of magnitude of $\sim h/$length scale), then we have:

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}\tau^2} = \frac{1}{2}h_{00,i}, \tag{19}$$

and we see that $-\frac{1}{2}h_{00}$ plays the role of the regular Newtonian gravitational potential. Thus spacetime that is flat with some slight curvature can produce an effect that is equivalent to Newtonian gravity. The deviation of the 00 part of the metric tensor from -1 is the regular gravitational potential.

Note that since $g_{\alpha\beta}$ has 10 independent components, there are 10 potentials; we will see later what the other 9 mean physically. Also note that a change of coordinates can change h_{00}, so in GR – like in Newtonian gravity – there are no absolute measures of the potential. We will study soon how locally measurable quantities relate to $h_{\alpha\beta}$.

IV. GEODESICS DEFINED BY THE VARIATIONAL PRINCIPLE

An alternative method to define a geodesic is by the variational principle: we can write $S = -m\int \mathrm{d}\tau$ and set $\delta S = 0$. Our objective here is to (a) show that this "variational" definition of the geodesic is equivalent to the "parallel transport" definition we introduced earlier.

Let's go ahead and write S as follows –

$$S = \int f\,\mathrm{d}\sigma = \int -m\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}}\,\mathrm{d}\sigma, \tag{20}$$

and use the canonical procedure to find the conjugate momentum:

$$\begin{aligned}
p_\gamma &= \frac{\partial f}{\partial(\mathrm{d}x^\gamma/\mathrm{d}\sigma)} \\
&= -m\frac{1}{2\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}}}(-)g_{\mu\nu}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\delta^\nu_\gamma + \delta^\mu_\gamma\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}\right) \\
&= m\frac{1}{\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}}}g_{\mu\gamma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \\
&= mg_{\mu\gamma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}, \\
&= mu_\gamma,
\end{aligned} \tag{21}$$

where we used the fact that the denominator is $\mathrm{d}\tau/\mathrm{d}\sigma$. We may then say

$$\frac{\mathrm{d}p_\gamma}{\mathrm{d}\sigma} = \frac{\partial f}{\partial x^\gamma} = -m\frac{-g_{\mu\nu,\gamma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}}{2\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}}} = \frac{1}{2}mg_{\mu\nu,\gamma}u^\mu u^\nu\frac{\mathrm{d}\tau}{\mathrm{d}\sigma}, \tag{22}$$

so

$$\frac{\mathrm{d}u_\gamma}{\mathrm{d}\tau} = \frac{1}{2}g_{\mu\nu,\gamma}u^\mu u^\nu. \tag{23}$$

This is the same as Eq. (14), except for the very last line (which depended on symmetry). So a geodesic defined by the variational principle is the same as that defined by parallel transport.

Depending on the circumstances, one or the other method of computing geodesics may be easier in practice.

A. Application to slow-moving particles

Let's now consider the variational principle for a slow-moving particle. We say that the particle moves through spacetime along a path $x^i(t)$ (yes, I will now use t as the parameter) with velocities $v^i(t) \equiv \mathrm{d}x^i/\mathrm{d}t$. I will also assume a weak gravitational field,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}. \tag{24}$$

Now the action is – keeping terms of order h and v^2, but dropping h^2, hv, and anything higher order:

$$\begin{aligned}
S &= -m\int \frac{\mathrm{d}\tau}{\mathrm{d}t}\,\mathrm{d}t \\
&= -m\int \sqrt{-g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t}}\,\mathrm{d}t \\
&= -m\int \sqrt{-g_{00} - 2g_{0i}v^i - g_{ij}v^iv^j}\,\mathrm{d}t \\
&= -m\int \sqrt{1 - h_{00} - 2h_{0i}v^i - \delta_{ij}v^iv^j - h_{ij}v^iv^j}\,\mathrm{d}t \\
&\approx -m\int \sqrt{1 - h_{00} - \delta_{ij}v^iv^j}\,\mathrm{d}t \\
&\approx -m\int \left(1 - \frac{1}{2}h_{00} - \frac{1}{2}\delta_{ij}v^iv^j\right)\mathrm{d}t \\
&\approx \int \left(-m + \frac{1}{2}mh_{00} + \frac{1}{2}m\delta_{ij}v^iv^j\right)\mathrm{d}t.
\end{aligned} \tag{25}$$

In this equation, the "$-m$" term is a constant and does nothing. The $\frac{1}{2}m\delta_{ij}v^iv^j$ term is the usual kinetic energy term if we write $S = \int(T-V)\,\mathrm{d}t$. The $\frac{1}{2}mh_{00}$ term is the potential energy term if we identify the gravitational potential as $\phi = -\frac{1}{2}h_{00}$. Thus once again, we see that ordinary Newtonian gravity can be derived by considering geodesics in the appropriate limit.

Note that what was "potential energy" in Newtonian physics is however just geometry in GR: in going from $\mathcal{A}$ to $\mathcal{B}$, a particle takes a path through higher gravitational potential because that path has a longer time, and it has a longer time because spacetime is stretched.

B. Gravitaitonal redshift, again

Finally, we consider the gravitational redshift. We learned by considering accelerating reference frames in special relativity that a clock that is higher in a gravitational field ticks at a faster rate in accordance with

$$\frac{\Delta\tau_{\mathrm{high}} - \Delta\tau_{\mathrm{low}}}{\Delta t} = [\text{gravitational field}] \times [\text{height difference}] = \phi_{\mathrm{high}} - \phi_{\mathrm{low}}. \tag{26}$$

This is consistent with having warped spacetime: the rate of ticking of a stationary clock is

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \sqrt{-g_{00}} = \sqrt{1-h_{00}} \approx 1 - \frac{1}{2}h_{00}, \tag{27}$$

if once again we identify $-\frac{1}{2}h_{00} = \phi$. Of course, this had to work for the description of gravity as curved spacetime to be successful.