Lecture VII: Particles, fluids and the stress-energy tensor

(Dated: September 11, 2019)

I. OVERVIEW

This lecture covers the key material from Chapter 4 of the book. This includes a discussion of particles and fluids in relativity, and most importantly the stress-energy tensor $T^{\mu\nu}$, which will be the source for gravity. I will use notation consistent with Table 4.1 of the book.

II. NUMBER FLUXES

A. Definitions

Let's begin by supposing we have a medium containing a conserved set of particles – we'll use baryons for concreteness (this is also the most relevant case in astrophysics). As these particles move through spacetime, the number $\mathcal{N}$ of baryons passing through a 3-dimensional hypersurface $\mathcal{V}$ must be written as an integral over that surface:

$$\mathcal{N} = \int_{\mathcal{V}} \boldsymbol{H} = \sum \frac{1}{3!} H_{\alpha\beta\gamma} \Delta V^{\alpha\beta\gamma} = \sum (H_{123}\Delta V^{123} + H_{023}\Delta V^{023} + H_{031}\Delta V^{031} + H_{012}\Delta V^{012}), \tag{1}$$

where $\boldsymbol{H}$ is a 3-form since the volume element $\Delta V^{\alpha\beta\gamma}$ is fully antisymmetric. We can see that by placing the surface in different directions, we can get a physical interpretation of the components of $\boldsymbol{H}$. For example, if we place the volume element in the xyz-hyperplane, so that $\Delta V^{123} = \Delta x \Delta y \Delta z$ and the components that are not permutations of 123 are zero, then we have

$$\mathcal{N} = \sum H_{123}\, \Delta x\, \Delta y\, \Delta z. \tag{2}$$

Thus H_{123} is the baryon number density (units: baryons per unit volume). Similarly, we may place the volume $\mathcal{V}$ in the tyz-hyperplane: then $\Delta V^{023} = \Delta t \Delta y \Delta z$, and we are now asking how many particles crossed a small amount of area in a given amount of time:

$$\mathcal{N} = \sum H_{023}\, \Delta t\, \Delta y\, \Delta z. \tag{3}$$

So then we see that $-H_{023}$ is a flux in the x-direction (units: baryons per unit area per unit time).

[Note I'm putting a $-$ sign here for consistency of orientation. I want to count a particle going through a surface with xyz orientation with positive t as a positive contribution to the number of particles passing through $\mathcal{V}$. Since $xyz; t$ is an odd permutation of $tyz; x$, I have to count a particle passing through tyz in the positive x direction as a negative contribution. This is sometimes the hardest part to figure out when dealing with handedness.]

A common way to express these ideas is with the dual: we write $\boldsymbol{H} = -{}^\star\boldsymbol{N}$, where $\boldsymbol{N}$ is a vector. Then:

$$[\text{number density}] = H_{123} = -\epsilon_{123\alpha} N^\alpha = -\epsilon_{1230} N^0 = N^0 \tag{4}$$

and

$$[x\text{−number flux}] = -H_{023} = \epsilon_{023\alpha} N^\alpha = \epsilon_{0231} N^1 = N^1. \tag{5}$$

In general, $\boldsymbol{N}$ can be thought of as the *number flux vector*; its time component is the density. The total number of particles crossing a hypersurface $\mathcal{V}$ is

$$\mathcal{N} = \int_{\mathcal{V}} -{}^\star\boldsymbol{N}. \tag{6}$$

We will write things in terms of $\boldsymbol{N}$ from now on, since 1 index is easier to deal with than 3.

B. Baryon number conservation

An important property of baryon number is that it is conserved. In the context of relativity, this means that if we have two 3D hypersurfaces $\mathcal{V}_\mathrm{A}$ and $\mathcal{V}_\mathrm{B}$ with the same boundary $\mathcal{S}$, then the number of particles crossing them is the same. You can think of global conservation as being the case where $\mathcal{V}_\mathrm{A}$ and $\mathcal{V}_\mathrm{B}$ fill all of space but at different times, and $\mathcal{S}$ is at ∞. The statement that the number of particle crossings is the same means that $\mathcal{N}_\mathrm{A} = \mathcal{N}_\mathrm{B}$, where

$$\mathcal{N}_\mathrm{A} = \int_{\mathcal{V}_\mathrm{A}} -{}^\star\boldsymbol{N} \quad \mathcal{N}_\mathrm{B} = \int_{\mathcal{V}_\mathrm{B}} -{}^\star\boldsymbol{N}. \tag{7}$$

Now since the two hypersurfaces share a boundary, we may think of them as enclosing a region $\mathcal{K}$ of 4D spacetime. Then the boundary $\partial\mathcal{K}$ consists of $\mathcal{V}_\mathrm{B}$ (at the later time) plus $\mathcal{V}_\mathrm{A}$ (at the earlier time) with the opposite orientation. That is,

$$\int_{\partial\mathcal{K}} -{}^\star\boldsymbol{N} = \int_{\mathcal{V}_\mathrm{B}} -{}^\star\boldsymbol{N} - \int_{\mathcal{V}_\mathrm{A}} -{}^\star\boldsymbol{N} = \mathcal{N}_\mathrm{B} - \mathcal{N}_\mathrm{A}. \tag{8}$$

The left-hand side can also be simplified using Gauss's divergence theorem:

$$\int_{\partial\mathcal{K}} -{}^\star\boldsymbol{N} = \int_{\mathcal{K}} -\tilde{\boldsymbol{d}}{}^\star\boldsymbol{N} = \int_{\mathcal{K}} (\boldsymbol{\nabla}\cdot\boldsymbol{N})\boldsymbol{\varepsilon} = \int_{\mathcal{K}} (N^\alpha{}_{,\alpha})\boldsymbol{\varepsilon}. \tag{9}$$

This means that

$$\mathcal{N}_\mathrm{B} - \mathcal{N}_\mathrm{A} = \int_{\mathcal{K}} (N^\alpha{}_{,\alpha})\boldsymbol{\varepsilon} = \int_{\mathcal{K}} (N^\alpha{}_{,\alpha})\,\mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3 \tag{10}$$

(the last two integrals mean the same thing, but the final notation corresponds to how you did volume integrals in multivariable calculus). Conservation of particles occurs for all such volumes $\mathcal{K}$ if and only if $\boldsymbol{N}$ is divergenceless:

$$N^\alpha{}_{,\alpha} = 0. \tag{11}$$

In special relativity, the total number of baryons in the Universe is obtained by taking a slice $\mathcal{V}$ through all of space at a particular time t:

$$\mathcal{N}_\mathrm{tot}(t) = \int_{\mathbb{R}^3} N^0(t,x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{12}$$

This is independent of t.

C. The momentarily comoving reference frame

As long as baryons are present somewhere, and their flux $\boldsymbol{N}$ is a timelike vector (true if baryons are real massive particles moving on timelike trajectories), then we may define a *momentarily comoving reference frame* (MCRF) as a frame whose 4-velocity $\boldsymbol{U}$ is parallel to $\boldsymbol{N}$:

$$\boldsymbol{N} = n\boldsymbol{U}, \quad \boldsymbol{U}\cdot\boldsymbol{U} = -1, \quad U^0 > 0. \tag{13}$$

Here n is the baryon number density that would be measured by an observer in the MCRF. It is distinct from the number density that would be measured in the lab frame, which is

$$N^0 = nU^0 = n\gamma, \tag{14}$$

where $\gamma = 1/\sqrt{1-v^2}$ and v is the 3-velocity of the MCRF. Note that the lab frame number density is always greater than or equal to the MCRF number density, with equality holding if $v = 0$ (i.e., the MCRF and lab frames coincide).

III. MOMENTUM AND ENERGY CONSERVATION

In flat spacetime, each component of total 4-momentum P^μ is conserved. Just as we could write the 4 components N^α of baryon number flux, we can similarly write the $4 \times 4 = 16$ components of 4-momentum flux $T^{\mu\alpha}$. These components form what is called the *stress-energy tensor*. The components are interpreted as follows:

- T^{00} is the energy density (conventionally J m^{-3}).
- T^{0j} is the energy flux in the j-direction.
- T^{i0} is the i component of the momentum density.
- T^{ij} is the flux of i component of momentum in the j direction. (This is known as the “stress tensor” in 3 dimensions, and should be very familiar to your friends in mechanical engineering.)

The globally conserved energy is

$$E(t) = \int_{\mathbb{R}^3} T^{00}(t, x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z, \tag{15}$$

and the globally conserved momentum is

$$P^i(t) = \int_{\mathbb{R}^3} T^{i0}(t, x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z. \tag{16}$$

The stress-energy tensor transforms as a 2nd rank tensor because both the momentum vector index μ and the flux vector index α in $T^{\mu\alpha}$ transform as vector indices:

$$T^{\bar{\mu}\bar{\alpha}} = \Lambda^{\bar{\mu}}{}_{\nu} \Lambda^{\bar{\alpha}}{}_{\beta} T^{\nu\beta}. \tag{17}$$

The local conservation law for energy and momentum can be written as

$$T^{\mu\alpha}{}_{,\alpha} = 0. \tag{18}$$

Remember this is actually 4 equations.

A. Symmetry

An important aspect of the stress-energy tensor is its symmetry. Newton’s generalized 3rd law says that two particles push on each other with a force acting along the line between them. The “along the line between them” was key to showing conservation of angular momentum. If that direction between them was given by a 3-vector $\boldsymbol{n}$, then the momentum exchanged between the particles and the direction that momentum is moving are both in direction $\boldsymbol{n}$; thus, $T^{ij} \propto n^i n^j$, and thus the contribution of this force to the stress tensor is symmetric. We can then conclude that all forces consistent with Newton’s 3rd law will have a symmetric stress tensor. In special relativity, if the 3×3 sub-block T^{ij} is symmetric in all reference frames, then the transformation law (Eq. 17) requires the full tensor to be symmetric:

$$T^{\mu\nu} = T^{\nu\mu}. \tag{19}$$

You might object that magnetic forces don’t actually act on a line between the particles. It turns out, though, that the stress in a magnetic field is symmetric. More generally, we will see that a force that comes from a Lagrangian in curved spacetime must have a symmetric stress-energy tensor.

IV. DESCRIPTION IN THE MCRF FOR A PERFECT FLUID

Let’s now consider an observer $\bar{\mathcal{O}}$ who is in the MCRF for a perfect fluid. In this frame, moving with the baryonic material at velocity $\boldsymbol{U}$ with components $(1, 0, 0, 0)$, the momentum flux is equal to the pressure p on all 3 axes, and the energy density is ρ. There is no energy flux in the MCRF for a perfect fluid (“energy flux in the baryon rest frame” would normally mean energy transport by conduction, radiation, or – if the MCRF is defined by coarse-graining a turbulent fluid – convection). Then:

$$\mathbf{T} \underset{\bar{\mathcal{O}}}{\longrightarrow} \left(\begin{array}{c|ccc} \rho & 0 & 0 & 0 \\ \hline 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{array}\right). \tag{20}$$

One can write this as

$$T^{\bar{\mu}\bar{\alpha}} = \rho U^{\bar{\mu}} U^{\bar{\alpha}} + p(\eta^{\bar{\mu}\bar{\alpha}} + U^{\bar{\mu}} U^{\bar{\alpha}}). \tag{21}$$

Since the objects involved are tensors, if this is true in the MCRF, it is true in a general frame:

$$T^{\mu\alpha} = \rho U^{\mu} U^{\alpha} + p(\eta^{\mu\alpha} + U^{\mu} U^{\alpha}). \tag{22}$$

In this class, I will mainly deal with perfect fluids when we try to solve problems in GR. This is mostly for simplicity, and because most problems we encounter will involve perfect fluids.

We can write the equations of motion for a perfect fluid by using Eq. (18) and plugging in Eq. (22). Defining the *enthalpy density* $h \equiv \rho + p$, we get

$$\begin{aligned} 0 = T^{\mu\alpha}{}_{,\alpha} &= \partial_\alpha[\rho U^\mu U^\alpha + p(\eta^{\mu\alpha} + U^\mu U^\alpha)] \\ &= \partial_\alpha[h U^\mu U^\alpha + p\eta^{\mu\alpha}] \\ &= h_{,\alpha} U^\mu U^\alpha + h U^\mu{}_{,\alpha} U^\alpha + h U^\mu U^\alpha{}_{,\alpha} + p_{,\alpha}\eta^{\mu\alpha}. \end{aligned} \tag{23}$$

Equation (23), and the corresponding equation for baryon density conservation

$$0 = \partial_\alpha(n U^\alpha) = n U^\alpha{}_{,\alpha} + n_{,\alpha} U^\alpha, \tag{24}$$

are 5 first-order PDEs (each has one time derivative) describing the evolution of 7 variables: n, ρ, p, and the 4 components of U^α. This means we need 2 more equations to define the behavior of a perfect fluid. One of these, of course, is the normalization $\eta_{\mu\nu} U^\mu U^\nu = -1$. The other is the *equation of state*: this is an algebraic (no derivatives) relation between n, ρ, and p, that depends on the type of fluid we are considering. The relation between baryon density, energy density, and pressure cannot be solved from GR alone, but depends on the microphysics of the fluid itself; in the case of neutron star matter, for example, it is a subject for nuclear physics.