Lecture V: Tensor algebra in flat spacetime

(Dated: September 4, 2019)

I. OVERVIEW

This lecture covers most of Chapter 3 (some of the algebra in Chapter 3 was covered in Lecture IV; the calculus I will leave for Lecture VI). The focus is on developing our notion of tensors and manipulation of tensors.

II. 1-FORMS

Recall that when we studied momentum, we ended up with an object that has a different transformation rule than a vector:

$$p_{\bar{\delta}} = [\mathbf{\Lambda}^{-1}]^{\gamma}{}_{\bar{\delta}} p_\gamma \quad \text{vs.} \quad x^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}{}_\beta x^\beta. \tag{1}$$

Such an object is called a 1-form. A 1-form can be written in terms of basis 1-forms:

$$\tilde{\boldsymbol{p}} = p_\gamma \tilde{\boldsymbol{\omega}}^\gamma, \tag{2}$$

where I have written a tilde to specifically denote that I want to treat the object on the left side as a 1-form rather than a vector. The basis 1-forms have components that are mostly zeros except for a single 1:

$$\tilde{\boldsymbol{\omega}}^0 \underset{\mathcal{O}}{\rightarrow} (1,0,0,0), \quad \tilde{\boldsymbol{\omega}}^1 \underset{\mathcal{O}}{\rightarrow} (0,1,0,0), \quad \tilde{\boldsymbol{\omega}}^2 \underset{\mathcal{O}}{\rightarrow} (0,0,1,0), \quad \text{and} \quad \tilde{\boldsymbol{\omega}}^3 \underset{\mathcal{O}}{\rightarrow} (0,0,0,1). \tag{3}$$

The product of a 1-form and a vector is a scalar and is invariant:

$$p_{\bar{\alpha}} x^{\bar{\alpha}} = ([\mathbf{\Lambda}^{-1}]^{\gamma}{}_{\bar{\alpha}} p_\gamma)(\Lambda^{\bar{\alpha}}{}_\beta x^\beta) = ([\mathbf{\Lambda}^{-1}]^{\gamma}{}_{\bar{\alpha}} \Lambda^{\bar{\alpha}}{}_\beta) p_\gamma x^\beta = \delta^\gamma_\beta p_\gamma x^\beta = p_\beta x^\beta. \tag{4}$$

Such combinations, with one index down and one index up, are often called *contractions*.

Note that the book – and most rigorous mathematical texts on differential geometry – **define** a 1-form based on this property. That is, given a vector space $\mathcal{V}$, they define a 1-form $\tilde{\boldsymbol{p}}$ to be a linear mapping $\tilde{\boldsymbol{p}} : \mathcal{V} \rightarrow \mathbb{R}$. In this view, one can write

$$\tilde{\boldsymbol{p}}(\boldsymbol{x}) = \sum_\beta p_\beta x^\beta, \tag{5}$$

where the left-hand side is fundamental and the right-hand side is based on the existence of components of the vector. The space of 1-forms is itself a vector space and is written $\mathcal{V}^\star$. I will take the view (more common in particle physics) that the transformation properties are fundamental. As long as you are dealing with vectors and tensors in a finite number of dimensions, the two views are equivalent.

III. ARBITRARY TENSORS

A tensor of rank $\binom{m}{n}$ is an object T with m up-indices and n down-indices, that transforms as follows:

$$T^{\bar{\alpha}_1\bar{\alpha}_2...\bar{\alpha}_m}{}_{\bar{\mu}_1\bar{\mu}_2...\bar{\mu}_n} = \Lambda^{\bar{\alpha}_1}{}_{\beta_1}\Lambda^{\bar{\alpha}_2}{}_{\beta_2}...\Lambda^{\bar{\alpha}_m}{}_{\beta_m}[\mathbf{\Lambda}^{-1}]^{\nu_1}{}_{\bar{\mu}_1}[\mathbf{\Lambda}^{-1}]^{\nu_2}{}_{\bar{\mu}_2}...[\mathbf{\Lambda}^{-1}]^{\nu_n}{}_{\bar{\mu}_n} T^{\beta_1\beta_2...\beta_m}{}_{\nu_1\nu_2...\nu_n}. \tag{6}$$

For example:

- A vector with components v^α is a $\binom{1}{0}$ tensor.
- A 1-form with components p_α is a $\binom{0}{1}$ tensor.
- The metric tensor $\eta_{\mu\nu}$ is a $\binom{0}{2}$ tensor.
- A scalar f is a $\binom{0}{0}$ tensor.

In D dimensions, a tensor has D^{m+n} components. The concepts of addition $\mathbf{T}+\mathbf{U}$ (of tensors of the same rank) and scalar multiplication $\lambda\mathbf{T}$ have their obvious meanings.

A. Raising and lowering indices

Note that because indices can be raised or lowered (see Lecture IV), as long as one has a metric tensor ($\eta_{\mu\nu}$ in special relativity), only the total rank $m+n$ of a tensor matters. Individual indices can be raised and lowered, e.g.,

$$T_\alpha{}^\beta{}_\gamma = \eta^{\beta\delta} T_{\alpha\delta\gamma} \quad \leftrightarrow \quad T_{\alpha\delta\gamma} = \eta_{\delta\beta} T_\alpha{}^\beta{}_\gamma. \tag{7}$$

However, it may commonly occur that the "up" or "down" case is more naturally associated with the definition of a tensor. For example, the concept of "displacement" exists even if you haven't defined an inner product, and it is naturally a vector Δx^α. I will highlight some of the results below that don't depend on inner products, in which case it is important what is up and what is down. However, for the physics applications in this class, the metric always exists.

B. Symmetry of indices

You know in linear algebra that some matrices $\mathbf{M}$ can be symmetric ($M_{\alpha\beta} = M_{\beta\alpha}$) or antisymmetric ($M_{\alpha\beta} = -M_{\beta\alpha}$). The same concept applies to tensors of rank 2 and higher – we may say that a tensor $\mathbf{T}$ is symmetric or antisymmetric in 2 indices. If a tensor is symmetric or antisymmetric in all of its indices, we call it *fully symmetric* or *fully antisymmetric*.

In D dimensions, where there are only D possible values of the indices, a fully antisymmetric tensor can have rank at most D. More generally, a tensor of rank k that is fully symmetric or fully antisymmetric has not D^k independent entries but rather

$$N_{\rm comp,sym} = \frac{(D+k-1)!}{k!(D-1)!} \quad \text{or} \quad N_{\rm comp,antisym} = \frac{D!}{k!(D-k)!} \tag{8}$$

(a good combinatorics exercise). A fully antisymmetric tensor of rank k with all lower indices (i.e., rank $\binom{0}{k}$) is called a *k-form* (a 0-form is a scalar and a 1-form is a covector).

Sometimes we want to write the symmetric or antisymmetric part of a tensor (just as we do for a matrix). We use parentheses to indicate symmetrization:

$$T_{(\alpha\beta)} = \frac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha}) \tag{9}$$

and brackets for antisymmetrization:

$$T_{[\alpha\beta]} = \frac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha}). \tag{10}$$

Symmetrizing or antisymmetrizing 3 or more indices$k$ indices results in a $k!$ in the denominator and $k!$ possible permutations of the indices (and in the case of antisymmetrization, the odd permutations of the indices get $-$ signs).

IV. MULTIPLYING TENSORS

Given two tensors $\mathbf{T}$ (rank $\binom{m_T}{n_T}$) and $\mathbf{U}$ (rank $\binom{m_U}{n_U}$), it is natural to ask whether they can be multiplied. There are, in fact, several ways to do this depending on the ranks of the tensors involved.

One type of product that always exists is the *outer product* or *tensor product*, which leads to a tensor of rank $\binom{m_T+m_U}{n_T+n_U}$. This tensor, sometimes denoted $\mathbf{T}\otimes\mathbf{U}$, has components:

$$[\mathbf{T}\otimes\mathbf{U}]^{\alpha_1...\alpha_{m_T}}{}_{\mu_1...\mu_{n_T}}{}^{\beta_1...\beta_{m_U}}{}_{\nu_1...\nu_{n_U}} = T^{\alpha_1...\alpha_{m_T}}{}_{\mu_1...\mu_{n_T}} U^{\beta_1...\beta_{m_U}}{}_{\nu_1...\nu_{n_U}}. \tag{11}$$

This object occurs frequently, but in index notation it is often easier to just write the right-hand side of this equation. Since scalars have rank $\binom{0}{0}$, scalar multiplication is actually a special case of the tensor product.

A. Contractions and the dot product

You know that the tensor product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ has rank $\binom{2}{0}$. In undergraduate physics, you probably learned about two ways of multiplying vectors – the dot product and the cross product – and you may be wondering how they relate to the tensor product. The answer in the case of the dot product is that if $\mathbf{S} = \boldsymbol{a} \otimes \boldsymbol{b}$, then

$$\boldsymbol{a} \cdot \boldsymbol{b} = \eta_{\alpha\beta} a^\alpha b^\beta = \eta_{\alpha\beta} S^{\alpha\beta} = S^\alpha{}_\alpha, \tag{12}$$

where at the end we lowered the second index on $\mathbf{S}$. The operation of giving two indices on a tensor the same label and summing over them is called a *trace* (in the matrix sense) or *contraction*. This is only allowed when one index is up and one index is down. The contraction reduces the rank of a tensor by 2; more precisely, it takes an $\binom{m}{n}$ tensor and returns an $\binom{m-1}{n-1}$ tensor. In this case, the operation of the tensor product, followed by index lowering, followed by contraction takes the $\binom{2}{0}$ rank $\mathbf{S}$ tensor, gives us a $\binom{1}{1}$ tensor, and ultimately produces a $\binom{0}{0}$ tensor (a scalar). In index notation, we will usually write $\eta_{\alpha\beta} a^\alpha b^\beta$ or $a^\alpha b_\alpha$.

The concept of contraction can be applied to tensors of higher rank, and the index notation naturally tells you which indices are used where.

B. The Levi-Civita tensor and the cross product

The other type of multiplication of vectors that you learned in undergraduate physics is the cross product. This is special for a number of reasons, but most important is that it has a sense of "handedness" that the dot product doesn't: you get opposite answers for right- or left-handed coordinate systems. I'll treat this in $D = 3$ dimensions first (hence with Latin indices), since that is the only case where you can take a cross product of two vectors and get a vector; we will then see what happens in $D = 4$ dimensions.

In 3-dimensional Euclidean space, we define the *Levi-Civita tensor* to be the rank 3 tensor that is fully antisymmetric and has

$$\varepsilon_{123} = 1. \tag{13}$$

Note that this completely defines $\boldsymbol{\varepsilon}$: because it is fully antisymmetric, components with repeated indices are zero (e.g., $\varepsilon_{322} = 0$), which leaves $3! = 6$ non-zero components. The remaining components are $+1$ if the 3 indices are in an even permutation of 123 (i.e., the coordinate axes in that order are right-handed) and -1 if they are an odd permutation of 123 (i.e., the coordinate axes in that order are left-handed):

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1 \quad \text{and} \quad \varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1. \tag{14}$$

You can show that under a rotation $\mathbf{R}$, the Levi-Civita tensor transforms according to, e.g.,

$$\varepsilon_{\bar{1}\bar{2}\bar{3}} = [\mathbf{R}^{-1}]^i{}_{\bar{1}} [\mathbf{R}^{-1}]^j{}_{\bar{2}} [\mathbf{R}^{-1}]^3{}_{\bar{k}} \varepsilon_{ijk} = \det(\mathbf{R}^{-1}). \tag{15}$$

(Yes, you can check that last part component-by-component: there is an implied sum over i, j, and k, with 6 non-trivial terms, 3 of each sign, and they are exactly the products of components of $\mathbf{R}^{-1}$ that you invoke to compute a determinant.) Since $\mathbf{R}$ has determinant 1, $\boldsymbol{\varepsilon}$ is invariant under proper rotations. (If you do an improper rotation and flip the handedness of your coordinate system, you get a $-$ sign.)

We can then write the cross product $\boldsymbol{w} = \boldsymbol{a} \times \boldsymbol{b}$ in 3 Euclidean dimensions by

$$w^i = \varepsilon^i{}_{jk} a^j b^k. \tag{16}$$

You can see that the cross product requires both a metric tensor (raising indices) and the Levi-Civita tensor. But also, the ability to multiply two vectors and get a third vector this way made essential use of $\boldsymbol{\varepsilon}$ having 3 indices, which was based on 3 dimensions.

In general, in D dimensions, we define the Levi-Civita tensor to be rank D and fully antisymmetric. In flat spacetime, we will set

$$\varepsilon_{0123} = 1, \tag{17}$$

so there are $4! = 24$ non-zero components of $\boldsymbol{\varepsilon}$, each of which is ± 1. Since Lorentz transformations have determinant 1, this tensor also is invariant under Lorentz transformations. But in 4 dimensions, if we want to write a cross product of vectors and get a vector, we will have to have 3 input vectors, e.g., $w^\alpha = \varepsilon^\alpha{}_{\beta\gamma\delta} a^\beta b^\gamma c^\delta$. Other possibilities, such as multiplying 2 vectors and getting a rank 2 antisymmetric tensor $\varepsilon^{\alpha\beta}{}_{\gamma\delta} a^\gamma b^\delta$, also exist. Most of the time, when dealing with the Levi-Civita tensor, we will just write in index notation which product we mean.

C. The wedge product and the dual

The above discussion suggests another generalization of the cross product. If we have a k-form $\boldsymbol{\alpha}$ and an ℓ-form $\boldsymbol{\beta}$, then we define the *wedge product* to be the $(k+\ell)$-form:

$$[\boldsymbol{\alpha} \wedge \boldsymbol{\beta}]_{\mu_1 \ldots \mu_k \nu_1 \ldots \nu_\ell} = \frac{(k+\ell)!}{k!\ell!} \alpha_{[\mu_1 \ldots \mu_k} \beta_{\nu_1 \ldots \nu_\ell]}, \tag{18}$$

where the encompassing bracket means we antisymmetrize over all $k+\ell$ indices. The pre-factor is a convention that ensures each of the $D!/(k!(D-k)!) \times D!/(\ell!(D-\ell)!)$ pairs of independent components of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ appears once (with either a $+$ or $-$ sign). The simplest example is the product of two 1-forms α_μ and β_ν:

$$[\boldsymbol{\alpha} \wedge \boldsymbol{\beta}]_{\mu\nu} = \frac{2!}{1!1!} \alpha_{[\mu} \beta_{\nu]} = \alpha_\mu \beta_\nu - \alpha_\nu \beta_\mu. \tag{19}$$

Note that the wedge product is associative, but satisfies

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = \begin{cases} \boldsymbol{\beta} \wedge \boldsymbol{\alpha} & \text{if } k \text{ or } \ell \text{ are even,} \\ -\boldsymbol{\beta} \wedge \boldsymbol{\alpha} & \text{if } k \text{ and } \ell \text{ areboth odd.} \end{cases} \tag{20}$$

We now define the *Hodge dual* (often simply the "dual") of a k-form $\boldsymbol{\alpha}$ to be the antisymmetric rank $D-k$ tensor given by

$$[{}^\star\boldsymbol{\alpha}]^{\mu_1 \ldots \mu_{D-k}} = \frac{1}{k!} \varepsilon^{\mu_1 \ldots \mu_{D-k} \nu_1 \ldots \nu_k} \alpha_{\nu_1 \ldots \nu_k}. \tag{21}$$

Thus for example, in 3-dimensional Euclidean space, an antisymmetric rank 2 tensor (antisymmetric matrix) has dual:

$$\mathbf{M} \underset{\mathcal{O}}{\rightarrow} \begin{pmatrix} 0 & M_{xy} & M_{xz} \\ -M_{xy} & 0 & M_{yz} \\ -M_{xz} & -M_{yz} & 0 \end{pmatrix} \quad \rightarrow \quad {}^\star\mathbf{M} \underset{\mathcal{O}}{\rightarrow} (M_{yz}, -M_{xz}, M_{xy}). \tag{22}$$

We can therefore write the cross product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as:

$$[\boldsymbol{a} \times \boldsymbol{b}]^i = \epsilon^i{}_{jk} a^j b^k = \frac{1}{2} \epsilon^{ijk} (a_j b_k - a_k b_j) \quad \rightarrow \quad \boldsymbol{a} \times \boldsymbol{b} = {}^\star(\boldsymbol{a} \wedge \boldsymbol{b}). \tag{23}$$

The concepts of dual and wedge product, unlike cross product, exist in any number of dimensions.

It can be shown that the Hodge dual acting twice gives back the original form:

$$^{\star\star}\boldsymbol{\alpha} = \pm\boldsymbol{\alpha}, \tag{24}$$

where the sign depends on the index raising/lowering and how many indices have to get switched when computing the dual twice. In Euclidean 3-space, we always have a $+$ sign; in 3+1-dimensional special relativity, we have a $-$ sign for even-rank $\boldsymbol{\alpha}$ and a $+$ sign for odd-rank $\boldsymbol{\alpha}$.