Lecture II: Examples from special relativity

(Dated: August 23, 2019)

I. OVERVIEW

This material covers some examples from special relativity. I will refer to them later, but they are mainly here to build your intuition and foreshadow applications we will discuss later in the course.

II. THOMAS PRECESSION

Problem: Suppose that students A and B start out in the lab frame. Student A remains in the lab frame. Student B gets in a rocket ship and changes their velocity by V_x in the x-direction (with $V_x \ll 1$) by firing the "+X" thrusters on their rocket. Then they change their velocity by V_y in the new y-direction (again with $V_y \ll 1$) by firing the "+Y" thrusters. Then they fire the "−X" thrusters (changing their velocity by $-V_x$), and the "−Y" thrusters (changing their velocity by $-V_y$). To lowest order in the ΔVs, how does B's reference frame now differ from A's?

Comment: The problem is designed so that in Newtonian physics, B's final reference frame is the same as A's: they are at rest with respect to each other, with no rotation of the coordinate systems.

Solution: We can write the Lorentz transformation matrix for going from A's frame (which is also B's initial frame) to B's final frame by multiplying the Lorentz transformation matrices. Remember that we apply matrices from right to left:

$$
\begin{aligned}
\mathbf{\Lambda} &= \mathbf{\Lambda}(-V_y\boldsymbol{e}_y)\mathbf{\Lambda}(V_x\boldsymbol{e}_x)\mathbf{\Lambda}(-V_y\boldsymbol{e}_y)\mathbf{\Lambda}(V_x\boldsymbol{e}_x) \\
&= \left(\begin{array}{c|ccc} \gamma_y & 0 & V_y\gamma_y & 0 \\ \hline 0 & 1 & 0 & 0 \\ V_y\gamma_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)
\left(\begin{array}{c|ccc} \gamma_x & V_x\gamma_x & 0 & 0 \\ \hline V_x\gamma_x & \gamma_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)
\left(\begin{array}{c|ccc} \gamma_y & 0 & -V_y\gamma_y & 0 \\ \hline 0 & 1 & 0 & 0 \\ -V_y\gamma_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)
\left(\begin{array}{c|ccc} \gamma_x & -V_x\gamma_x & 0 & 0 \\ \hline -V_x\gamma_x & \gamma_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \\
&= \left(\begin{array}{c|ccc} \gamma_x\gamma_y & V_x\gamma_x\gamma_y & V_y\gamma_y & 0 \\ \hline V_x\gamma_x & \gamma_x & 0 & 0 \\ V_y\gamma_x\gamma_y & V_xV_y\gamma_x\gamma_y & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)
\left(\begin{array}{c|ccc} \gamma_x\gamma_y & -V_x\gamma_x\gamma_y & -V_y\gamma_y & 0 \\ \hline -V_x\gamma_x & \gamma_x & 0 & 0 \\ -V_y\gamma_x\gamma_y & V_xV_y\gamma_x\gamma_y & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \\
&= \left(\begin{array}{c|ccc} \gamma_x^2\gamma_y^2 - V_x^2\gamma_x^2\gamma_y - V_y^2\gamma_x\gamma_y^2 & -V_x^2\gamma_x^2\gamma_y^2 + V_x^2\gamma_x^2\gamma_y + V_xV_y^2\gamma_x\gamma_y^2 & -V_y\gamma_x\gamma_y^2 + V_y\gamma_y^2 & 0 \\ \hline V_x\gamma_x^2\gamma_y - V_x\gamma_x^2 & -V_x^2\gamma_x^2\gamma_y + \gamma_x^2 & -V_xV_y\gamma_x\gamma_y & 0 \\ V_y\gamma_x^2\gamma_y^2 - V_x^2V_y\gamma_x^2\gamma_y - V_y\gamma_x\gamma_y^2 & -V_xV_y\gamma_x^2\gamma_y^2 + V_xV_y\gamma_x^2\gamma_y + V_xV_y\gamma_x\gamma_y^2 & -V_y^2\gamma_x\gamma_y^2 + \gamma_y^2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \\
&= \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & -V_xV_y & 0 \\ 0 & V_xV_y & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) + \text{order } V^3,
\end{aligned}
\tag{1}
$$

where we have used the Taylor expansions $\gamma_x = 1 + \frac{1}{2}V_x^2 + ...$ and $\gamma_x^2 = 1 + V_x^2 + ...$. This corresponds – to second order in the velocities – to a rotation around the z-axis by angle $-V_xV_y$.

Interpretation: This rotation of the coordinate system when an observer boosts and comes back to their original velocity is called *Thomas precession*. It has no analogue in Newtonian physics. In general, if an observer boosts around a curve in the (v_x, v_y)-plane with some area $\mathcal{A} = \oint v_x\,dv_y$, then the observer's reference frame rotates around z by angle $\mathcal{A}$.

One context where this is important is atomic physics: the spin of an electron orbiting an atom undergoes Thomas precession. A semiclassical description would be that as the electron orbits around many times, the direction of its spin angular momentum will precess around the direction of the orbital angular momentum. Since the typical velocity of an electron in a light atom is $1/137$ (the first Bohr orbit velocity in units where $c = 1$), we expect the precession to be by an angle of $1/137^2$ per orbit, or equivalently the precession frequency should be $1/(2\pi \times 137^2) \sim 8 \times 10^{-6}$ of the orbital frequency – that is, about 50 GHz. This is a contribution to the fine structure splitting of spectral lines or energy levels of atoms (11 GHz in hydrogen 2p; as one moves down the periodic table, the splittings are larger), but note that magnetic interactions give contributions that are the same order of magnitude.

The other context in which we care about Thomas precession is for space dynamics (satellites carrying gyroscopes orbiting the Earth, or binary stars). In the case of orbiting the Earth at a velocity of 7.5 km/s, or $2.5 \times 10^{-5}c$, we expect that each orbit yields a change in angle of

$$\pi \times (2.5 \times 10^{-5})^2 = 2 \times 10^{-9}\,\text{radians} = 0.4\,\text{milliarcsec.} \tag{2}$$

A correct calculation for an orbiting satellite requires general relativity – the result is a factor of 3 different. In this context the phenomenon is known as *geodetic precession.* The measurement of geodetic precession by Gravity Probe B is one of the tests of general relativity.

III. THE TWINS PARADOX

The standard version of this paradox involves two students, A and B, who are initially the same age. Student A remains on Earth, while Student B gets in a rocket ship and travels at a velocity V away from Earth in the x-direction. After some time T, and at distance $L = VT$ from Earth, Student B's rocket fires its engines, reverses its velocity (now $-V$ in the x-direction) and comes back to Earth, arriving at time $2T$ as measured by A. At this time, Student B's rocket fires its engines again and is thereafter at rest with respect to Earth.

[Note: the book devotes §1.13 to this problem, but I want to treat it a bit differently and talk about accelerated frames, to give you a preview of what has to happen later when we get to gravity.]

A. The setup

Problem: What is the difference between the age of A and the age of B?

Solution: We will denote the events as follows – $\mathcal{P}_1$ is the event where Student B leaves Earth, $\mathcal{P}_2$ is where Student B's rocket fires to come home, and $\mathcal{P}_3$ is Student B's triumphant return to their home planet.

From $\mathcal{P}_1$ to $\mathcal{P}_3$, Student A aged by $\Delta\tau_\text{A}(\mathcal{P}_1 \rightarrow \mathcal{P}_3) = 2T$. We must do a similar calculation for Student B. From $\mathcal{P}_1$ to $\mathcal{P}_2$, in A's frame, the change in coordinates were

$$\Delta x^\alpha = (T, VT, 0, 0), \tag{3}$$

and the proper time interval is

$$\Delta\tau_\text{B}(\mathcal{P}_1 \rightarrow \mathcal{P}_2) = \sqrt{\Delta t^2 - \Delta x^2} = \sqrt{T^2 - (VT)^2} = T\sqrt{1 - V^2}. \tag{4}$$

The proper time experienced by B on their way back to Earth ($\mathcal{P}_2 \rightarrow \mathcal{P}_3$) is the same: also $T\sqrt{1-V^2}$. Thus the total time elapsed as seen by B is

$$\Delta\tau_\text{B}(\mathcal{P}_1 \rightarrow \mathcal{P}_3) = 2T\sqrt{1 - V^2}, \tag{5}$$

and the difference in time is

$$\Delta\tau_\text{A}(\mathcal{P}_1 \rightarrow \mathcal{P}_3) - \Delta\tau_\text{B}(\mathcal{P}_1 \rightarrow \mathcal{P}_3) = 2T\left(1 - \sqrt{1 - V^2}\right). \tag{6}$$

This is positive, so at the end of the mission, Student A is older than Student B.

Comment: In what follows, it is most convenient to express this via Taylor expansion:

$$\Delta\tau_\text{A}(\mathcal{P}_1 \rightarrow \mathcal{P}_3) - \Delta\tau_\text{B}(\mathcal{P}_1 \rightarrow \mathcal{P}_3) = T\left(V^2 + \frac{1}{4}V^4 + ...\right) \approx TV^2. \tag{7}$$

B. What did Student B see?

A common question we get is that if all reference frames are the same, what the real difference is between what Student A sees and what Student B sees. In Student A's reference frame, of course, Student B appears to be aging slowly, by a factor of $\sqrt{1-V^2}$, the whole time. But it is a deeper question what Student B sees. During the outbound transit from $\mathcal{P}_1$ to $\mathcal{P}_2$, Student B must think that Student A is aging more slowly by a factor of $\sqrt{1-V^2}$. Thus, when Student B gets to $\mathcal{P}_2$, they think that Student A is **younger** than they are, by an amount

$$T\left(1-\sqrt{1-V^2}\right) \approx \frac{1}{2}TV^2. \tag{8}$$

Similarly, during the inbound transit from $\mathcal{P}_2$ to $\mathcal{P}_3$, Student B must think that Student A is aging more slowly than they are by a factor of $\sqrt{1-V^2}$, and again Student B thinks that Student A ages less than they do. So at the end, if one adds up the two legs of the journey, Student B thinks that Student A is younger than them by TV^2. But at the end of the mission, the two students are in the same reference frame and can compare their watches. So it can't be both that Student A is TV^2 older than Student B but also Student A is TV^2 younger than Student B; there must somehow be an additional contribution of $2TV^2$ in order for special relativity to be consistent. So where does this come from?

The answer has to do with what Student B did at $\mathcal{P}_2$: during the brief period of time when they were firing their rocket engines, they were not in an inertial frame. Instead, they were accelerating, with some acceleration g (convention: $g<0$, since the acceleration is toward $-x$) for time δt. I will treat this problem in the case of $V \ll 1$, so that I can approximate

$$g\,\delta t = -2V, \tag{9}$$

and treat the acceleration as instantaneous in the sense that $\delta t \ll T$.

I will do a treatment of accelerating frames only in the limit of velocities small compared to c, which is sufficient to get the point across. If I have an observer $\bar{\mathcal{O}}$ moving at velocity v relative to an inertial frame $\mathcal{O}$ in the x-direction, then to order v we have:

$$t = \bar{t} + v\bar{x}. \tag{10}$$

Now I want to make the barred observer accelerate: I thus write $v = g\bar{t}$. This implies

$$t = \bar{t} + g\bar{t}\bar{x} = (1+g\bar{x})\bar{t}. \tag{11}$$

(Note – this procedure doesn't work for computing x, because I would need to take into account that the origins of the coordinate systems don't have the same spatial coordinates. At order v, this complication doesn't enter into discussion of the time coordinates – that only happens at order v^2.) We can thus see that the rate of passage of time as measured by the inertial observer is different from that measured by the accelerated observer, **even if they instantaneously have the same velocity**. The ratio is

$$\frac{\delta t}{\delta \bar{t}} = 1 + g\bar{x}: \tag{12}$$

this means that for $g<0$ ($\bar{\mathcal{O}}$ accelerating to the left), then less time passes in regions that are to the right (large $\bar{x}$) than to the left (small $\bar{x}$). In particular, in Student B's accelerated frame, Student A is aging faster than Student B, by a factor of $\approx 1 - gVT$ since Student A is at a distance VT to the left of Student B. During an engine firing of duration δt, this means Student A ages more than Student B by an amount

$$-gVT \times \delta t = -VT(g\,\delta t) = -VT(-2V) = 2TV^2. \tag{13}$$

This solves the paradox (at least to order V^2; special relativity is consistent to all orders, but this gets the point across), but with an important physical consequence. The consequence is that accelerated observers see time passing at different rates at different heights (different positions $\bar{x}$, in the language of Eq. 12). Later on, we will postulate that acceleration is equivalent to being in a gravitational field; this postulate will force us to conclude that clocks tick at different rates depending on their height (a higher clock runs faster than a lower clock), and this will force us to describe gravity in terms of curved spacetime.

The difference of rate of passage of time as a function of height is very small by everyday standards: for a 100 m height difference, for example, and after 1 year, the difference in elapsed time between two clocks is – putting back the factor of c^2 –

$$ght = \frac{ght}{c^2} = \frac{(9.8\,\mathrm{m/s^2}) \times (100\,\mathrm{m}) \times (3\times 10^7\,\mathrm{s})}{(3\times 10^8\,\mathrm{m/s})^2} = 3\times 10^{-7}\,\mathrm{s} = 0.3\,\mu\mathrm{s}. \tag{14}$$

Nevertheless, the time difference induced by Earth's gravity is measurable with the precision of atomic clocks.