Physics 6820 – Homework 8 Solutions

1. Barrier penetration for a scalar wave. [28 points]
Let's consider a massless scalar wave arriving at a Schwarzschild black hole in the long-wavelength limit, $\omega M \ll 1$. The objective of this problem is to determine the cross section for this scalar wave to be absorbed by the black hole. Recall that the cross section for a wave to penetrate the barrier is given in partial wave scattering theory by

$$\sigma = \frac{\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\mathbb{T}_\ell, \tag{1}$$

where $\mathbb{T}_\ell$ is the power transmission coefficient, and k is the wavenumber.

(a) [4 points] *Explain qualitatively why for $\omega M \ll 1$ you expect the $\ell = 0$ term to dominate.*

The usual angular momentum barrier gives $U_\ell(\bar r) \approx \ell(\ell+1)/\bar r^2$ at $\bar r \gg M$, so for $\ell \geq 1$ this barrier results in a classically forbidden region at $\bar r < \sqrt{\ell(\ell+1)}\,\omega^{-1}$. So if $\omega^{-1} \gg M$, the higher angular momentum partial waves cannot (classically) penetrate to the vicinity of the black hole. This barrier is absent for $\ell = 0$.

In what follows, we will set up a barrier penetration problem:

$$\Psi_{00}(t,\bar r) = \begin{cases} (\mathrm{e}^{-\mathrm{i}\omega\bar r} + R_0\mathrm{e}^{\mathrm{i}\omega\bar r})\mathrm{e}^{-\mathrm{i}\omega t} & \bar r \to \infty \\ T_0\mathrm{e}^{-\mathrm{i}\omega\bar r}\mathrm{e}^{-\mathrm{i}\omega t} & \bar r \to -\infty \end{cases}, \tag{2}$$

where R_0 and T_0 are the amplitude reflection and transmission coefficients.

(b) [5 points] *Show that in the regime $|\bar r| \ll \omega^{-1}$, the equations from the notes imply*

$$r^2\left(1-\frac{2M}{r}\right)\partial_r\frac{\Psi_{00}}{r} = K, \tag{3}$$

*where K is a constant. [*Hint*: Start with Lecture XXI, Eq. (6), and argue that the time derivative can be dropped.]*

Lecture XXI, Eq. (6) reads:

$$-\frac{1}{1-2M/r}\partial_t^2\frac{\Psi_{\ell m}}{r} + \left(1-\frac{2M}{r}\right)\partial_r^2\frac{\Psi_{\ell m}}{r} + \left(\frac{2}{r}-\frac{2M}{r^2}\right)\partial_r\frac{\Psi_{\ell m}}{r} - \frac{\ell(\ell+1)}{r^2}\frac{\Psi_{\ell m}}{r} = 0. \tag{4}$$

In the $\ell = 0$ case, the last term goes away. Moreover, in the first term, $\partial_t^2 \to -\omega^2$.: Now in the vicinity of the barrier (small $\bar r$), we may neglect ω in comparison to the r derivatives, and then:

$$\left(1-\frac{2M}{r}\right)\partial_r^2\frac{\Psi_{\ell m}}{r} + \left(\frac{2}{r}-\frac{2M}{r^2}\right)\partial_r\frac{\Psi_{\ell m}}{r} = 0. \tag{5}$$

Using the product rule, the left-hand side can be re-written as

$$\frac{1}{r^2}\partial_r\left[r^2\left(1-\frac{2M}{r}\right)\partial_r\frac{\Psi_{00}}{r}\right]. \tag{6}$$

This being zero implies that the quantity in brackets is zero, hence Eq. (3).

(c) [5 points] *Show by integrating over r that the solution to part (b) is*

$$\Psi_{00} = \left[\frac{K}{2M}\ln\frac{r-2M}{r} + C\right]r, \tag{7}$$

where C is another constant.

Re-arranging Eq. (3) gives

$$\partial_r\frac{\Psi_{00}}{r} = \frac{K}{r^2(1-2M/r)}. \tag{8}$$

The integral is:

$$\frac{\Psi_{00}}{r} = \int\frac{K}{r^2(1-2M/r)}\,\mathrm{d}r = \int\frac{K}{r(r-2M)}\,\mathrm{d}r = \int\frac{K}{2M}\left(\frac{1}{r-2M}-\frac{1}{r}\right)\mathrm{d}r = \frac{K}{2M}\ln\frac{r-2M}{r} + C. \tag{9}$$

This leads to Eq. (7).

(d) [6 points] *Now go to small $\bar{r}$ (i.e., $r \approx 2M$) and force the solution to match onto Eq. (2). Show that the solution has $K \approx 0$, and express C in terms of T_0. By matching the solution at $2M \ll r \ll \omega^{-1}$, show that*

$$T_0 = -4\mathrm{i}\omega M. \tag{10}$$

On the left-hand side of the barrier, $r \approx 2M$ but $|\bar{r}| \ll \omega^{-1}$, we will have

$$\Psi_{00} = T_0 \mathrm{e}^{-\mathrm{i}\omega\bar{r}} \mathrm{e}^{-i\omega t} \approx T_0 \left[1 - \mathrm{i}\omega r - 2\mathrm{i}M\omega \ln \frac{r-2M}{2M}\right] \mathrm{e}^{-i\omega t}, \tag{11}$$

or – dividing by r and taking $r \approx 2M$ except in the logarithmic factors where the difference matters –

$$\frac{\Psi_{00}}{r} \approx T_0 \left[\frac{1}{2M} - \mathrm{i}\omega - \mathrm{i}\omega \ln \frac{r-2M}{2M}\right] \mathrm{e}^{-i\omega t}. \tag{12}$$

This looks like Eq. (7) with the matching:

$$K \approx -2\mathrm{i}M\omega T_0 \mathrm{e}^{-i\omega t} \quad \text{and} \quad C = \left(\frac{1}{2M} - \mathrm{i}\omega\right) T_0 \mathrm{e}^{-i\omega t}. \tag{13}$$

In the limit of $M\omega \ll 1$, this becomes

$$K \approx 0 \quad \text{and} \quad C \approx \frac{1}{2M} T_0 \mathrm{e}^{-i\omega t}. \tag{14}$$

Now taking the solution with these coefficients, Eq. (7) says

$$\Psi_{00} \approx \frac{r}{2M} T_0 \mathrm{e}^{-i\omega t}, \tag{15}$$

and at $2M \ll r \ll \omega^{-1}$, we can take $r \to \bar{r}$ and:

$$\Psi_{00} \approx \frac{\bar{r}}{2M} T_0 \mathrm{e}^{-i\omega t}. \tag{16}$$

This implies

$$\mathrm{e}^{-\mathrm{i}\omega\bar{r}} + R_0 \mathrm{e}^{\mathrm{i}\omega\bar{r}} \approx \frac{\bar{r}}{2M} T_0 \quad (\bar{r} \ll \omega^{-1}). \tag{17}$$

Taylor-expanding the left-hand side in this regime gives

$$(1 + R_0) - \mathrm{i}\omega(1 - R_0)\bar{r} \approx \frac{\bar{r}}{2M} T_0 \quad (\bar{r} \ll \omega^{-1}). \tag{18}$$

Matching of the 0th order term gives $R_0 = -1$, and then $1 - R_0 \approx 2$. Matching the 1st order term gives

$$-2\mathrm{i}\omega \approx \frac{1}{2M} T_0 \quad \to \quad T_0 \approx -4\mathrm{i}M\omega. \tag{19}$$

(e) [4 points] *Now again in the limit of $\omega M \ll 1$, find the power transmission coefficient $\mathbb{T}_0 = |T|^2$.*

Let's take the square norm:

$$\mathbb{T}_0 = |T_0|^2 = |-4\mathrm{i}M\omega|^2 = 16M^2\omega^2. \tag{20}$$

(f) [4 points] *Show that the cross section for the black hole to absorb an incident scalar wave in this limit is*

$$\sigma = 16\pi M^2. \tag{21}$$

How does this compare with the cross section $\pi b_{\rm crit}^2 = \pi(3\sqrt{3}\,M)^2 = 27\pi M^2$ that we derived for massless particles traveling on geodesics (which applies when ω is large and geometric optics can be used in place of wave optics)?

The cross section is now obtained from the partial wave formula, taking only the $\ell = 0$ term:

$$\sigma = \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell + 1)\mathbb{T}_\ell \approx \frac{\pi}{\omega^2} \mathbb{T}_0 = \frac{\pi}{\omega^2} \times 16M^2\omega^2 = 16\pi M^2. \tag{22}$$

This is 59% of the geometric optics formula, $27\pi M^2$.