Physics 6820 – Homework 8

(Dated: Due: November 22, 2019)

1. Barrier penetration for a scalar wave. [28 points]

Let's consider a massless scalar wave arriving at a Schwarzschild black hole in the long-wavelength limit, $\omega M \ll 1$. The objective of this problem is to determine the cross section for this scalar wave to be absorbed by the black hole. Recall that the cross section for a wave to penetrate the barrier is given in partial wave scattering theory by

$$\sigma = \frac{\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\mathbb{T}_\ell, \tag{1}$$

where $\mathbb{T}_\ell$ is the power transmission coefficient, and k is the wavenumber.

(a) [4 points] Explain qualitatively why for $\omega M \ll 1$ you expect the $\ell = 0$ term to dominate.

In what follows, we will set up a barrier penetration problem:

$$\Psi_{00}(t,\bar{r}) = \begin{cases} (\mathrm{e}^{-\mathrm{i}\omega\bar{r}} + R_0\mathrm{e}^{\mathrm{i}\omega\bar{r}})\mathrm{e}^{-\mathrm{i}\omega t} & \bar{r}\to\infty \\ T_0\mathrm{e}^{-\mathrm{i}\omega\bar{r}}\mathrm{e}^{-\mathrm{i}\omega t} & \bar{r}\to-\infty \end{cases}, \tag{2}$$

where R_0 and T_0 are the amplitude reflection and transmission coefficients.

(b) [5 points] Show that in the regime $|\bar{r}| \ll \omega^{-1}$, the equations from the notes imply

$$r^2\left(1-\frac{2M}{r}\right)\partial_r\frac{\Psi_{00}}{r} = K, \tag{3}$$

where K is a constant. [*Hint*: Start with Lecture XXI, Eq. (6), and argue that the time derivative can be dropped.]

(c) [5 points] Show by integrating over r that the solution to part (b) is

$$\Psi_{00} = \left[\frac{K}{2M}\ln\frac{r-2M}{r} + C\right]r, \tag{4}$$

where C is another constant.

(d) [6 points] Now go to small $\bar{r}$ (i.e., $r \approx 2M$) and force the solution to match onto Eq. (2). Show that the solution has $K \approx 0$, and express C in terms of T_0. By matching the solution at $2M \ll r \ll \omega^{-1}$, show that

$$T_0 = -4\mathrm{i}\omega M. \tag{5}$$

(e) [4 points] Now again in the limit of $\omega M \ll 1$, find the power transmission coefficient $\mathbb{T}_0 = |T|^2$.

(f) [4 points] Show that the cross section for the black hole to absorb an incident scalar wave in this limit is

$$\sigma = 16\pi M^2. \tag{6}$$

How does this compare with the cross section $\pi b_{\mathrm{crit}}^2 = \pi(3\sqrt{3}\,M)^2 = 27\pi M^2$ that we derived for massless particles traveling on geodesics (which applies when ω is large and geometric optics can be used in place of wave optics)?