Physics 6820 – Homework 7 Solutions

1. The view from an observer hovering just above the horizon. [28 points]

In Lecture XVIII, §IIE, we considered observer $\mathcal{B}$ who hovers just above the horizon, remaining stationary with respect to the black hole, at $r = 2M(1+\epsilon)$, with $\epsilon \ll 1$. We showed that if observer $\mathcal{B}$ looks upward, at an angle δ from the zenith, they can see the distant sky if

$$\delta < \alpha \approx \frac{3\sqrt{3}}{2}\sqrt{\epsilon}. \tag{1}$$

(The region outside radius α appears to be covered up by the black hole.) In class, I told you that if $\mathcal{B}$ looks at an angle δ with $0 < \delta < \alpha$ from the vertical, they can see constellations that are in reality a distance ψ from the zenith. Clearly $\psi = 0$ if $\delta = 0$, and as δ increases one reaches $\psi = \pi$ (the image of the constellations at nadir, but turned into a ring), then $\psi = 2\pi$ (the image of constellations at zenith, but also turned into a ring), etc. The objective of this problem is to determine the mapping between ψ and δ.

You should give your answers to this problem to lowest order in δ and ϵ.

(a) [4 points] *Assuming $\delta \ll 1$, show that the ratio $\tilde{\mathcal{L}}/\tilde{\mathcal{E}}$ of angular momentum to energy for the incoming photon is*

$$\frac{\tilde{\mathcal{L}}}{\tilde{\mathcal{E}}} = 3\sqrt{3}\,M\frac{\delta}{\alpha}. \tag{2}$$

In the orthonormal frame at observer $\mathcal{B}$, if the photon has energy $E_{\mathcal{B}}$, and we are working in the equatorial plane ($\theta = \pi/2$), then we have

$$p_{\hat\phi} = E_{\mathcal{B}}\sin\delta \approx E_{\mathcal{B}}\delta \quad \text{and} \quad -p_{\hat t} = E_{\mathcal{B}}, \tag{3}$$

so

$$\frac{p_{\hat\phi}}{-p_{\hat t}} = \delta. \tag{4}$$

Then the ratio of angular momentum to energy is

$$\frac{\tilde{\mathcal{L}}}{\tilde{\mathcal{E}}} = \frac{p_\phi}{-p_t} = \frac{(g_{\phi\phi})^{1/2}\,p_{\hat\phi}}{(-g_{tt})^{1/2}\,(-p_{\hat t})} = \frac{r}{\sqrt{1-2M/r}}\delta \approx \frac{2M}{\sqrt{\epsilon}}\delta \approx \frac{2M}{\sqrt{\epsilon}}\frac{\delta}{\alpha}\frac{3\sqrt{3}}{2}\sqrt{\epsilon} = 3\sqrt{3}\,M\frac{\delta}{\alpha}. \tag{5}$$

(b) [6 points] *Show that for a photon on an equatorial orbit (placing $\mathcal{B}$ over the equator for simplicity):*

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = \pm\frac{3\sqrt{3}\,M(\delta/\alpha)}{r^2\sqrt{1-27(\delta/\alpha)^2(M/r)^2(1-2M/r)}} \tag{6}$$

We may use the expression for u^r (Lecture XVIII) to write this as:

$$\begin{aligned}
\frac{\mathrm{d}\phi}{\mathrm{d}r} &= \frac{u^\phi}{u^r} \\
&= \pm\frac{\tilde{\mathcal{L}}/r^2}{-\sqrt{\tilde{\mathcal{E}}^2 - (1+\tilde{\mathcal{L}}^2/r^2)(1-2M/r)}} \\
&= \pm\frac{\tilde{\mathcal{L}}/r^2}{-\sqrt{\tilde{\mathcal{E}}^2 - (\tilde{\mathcal{L}}^2/r^2)(1-2M/r)}} \\
&= \pm\frac{\tilde{\mathcal{L}}/\tilde{\mathcal{E}}}{r^2\sqrt{1-(\tilde{\mathcal{L}}/\tilde{\mathcal{E}})^2(1-2M/r)/r^2}} \\
&= \pm\frac{3\sqrt{3}\,M(\delta/\alpha)}{r^2\sqrt{1-27(\delta/\alpha)^2(M/r)^2(1-2M/r)}}.
\end{aligned} \tag{7}$$

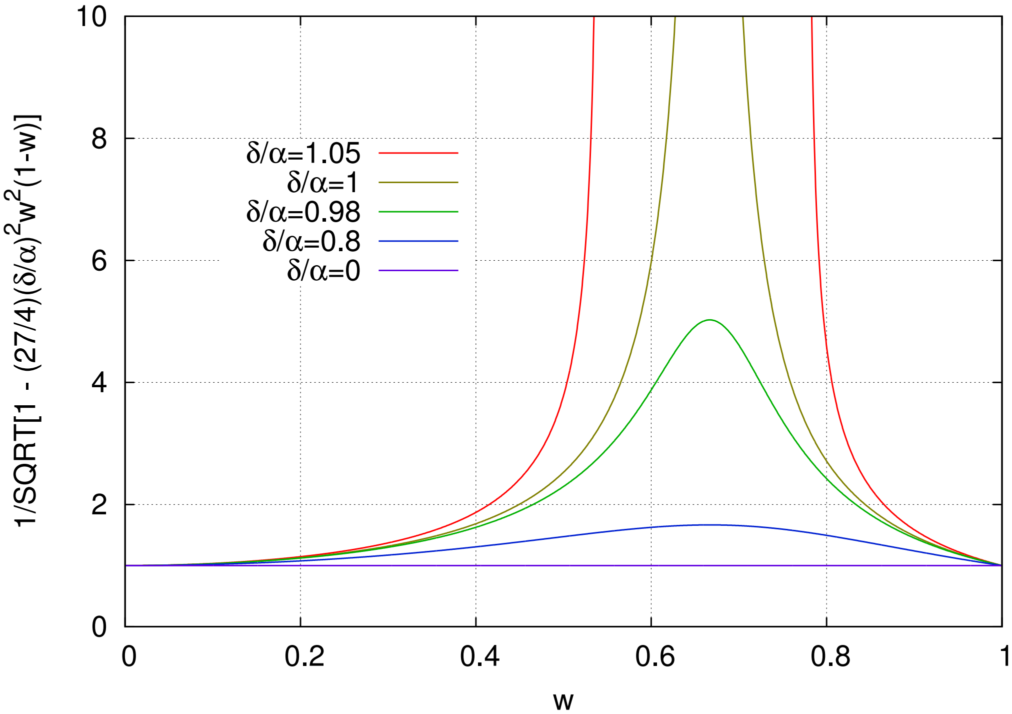

FIG. 1: The integrand in Eq. (9) for different values of δ/α.

In the third line, we took the limit that the mass goes to zero, i.e., $\tilde{\mathcal{L}}, \tilde{\mathcal{E}} \to \infty$, so we can drop the 1. In the last line, we substituted the result from part (a).

(c) [10 points] *Express ψ in terms of an integral over part (b), and make a plot of ψ vs. δ/α. Your result should not depend explicitly on ϵ as long as ϵ is very small. (You may want to do the integral numerically.)*

We want to find the longitude change ψ for the photon, running from its location just above the horizon to ∞. That is,

$$\psi = \int_{2M(1+\epsilon)}^{\infty} \frac{\mathrm{d}\phi}{\mathrm{d}r}\,\mathrm{d}r = \int_{2M(1+\epsilon)}^{\infty} \frac{3\sqrt{3}\,M(\delta/\alpha)}{r^2\sqrt{1-27(\delta/\alpha)^2(M/r)^2(1-2M/r)}}\,\mathrm{d}r. \tag{8}$$

It is useful to change variables to $w = 2M/r$:

$$\psi = \frac{3\sqrt{3}}{2}\frac{\delta}{\alpha}\int_0^1 \frac{1}{\sqrt{1-\frac{27}{4}(\delta/\alpha)^2 w^2(1-w)}}\,\mathrm{d}w. \tag{9}$$

At this point, we take the upper limit to 1 since it is clear that the integrand is well-behaved as $w \to 1$ (i.e., as one approaches the horizon). The maximum of this function is where the denominator is minimized or where $f(w) = \frac{27}{4}w^2(1-w)$ is maximized. By setting the derivative to zero, we see that $f(w)$ is maximized when $w = \frac{2}{3}$, and at that point $f(w) = 1$. So we can see that so long as $\delta/\alpha < 1$, the integrand is well-behaved everywhere. At $\delta/\alpha = 1$, the denominator goes to zero at $w = \frac{2}{3}$ and the integral will diverge. For $\delta/\alpha > 1$, there is a region near $w = \frac{2}{3}$ where the argument of the square root is negative, and the integral does not make sense (the photon cannot traverse that region, so it does not make sense to talk about its change of longitude). Examples of this behavior are shown in Fig. 1.

There are a few ways to compute the integral. I'll use a numerical method. The perl script that generates the plot is:

```
open(OUT, "|tee datadump-7c.dat|gnuplot") or die;
print OUT "set term postscript enhanced 20 eps color\n";
print OUT qq|set output "7c.eps"\n|;
```

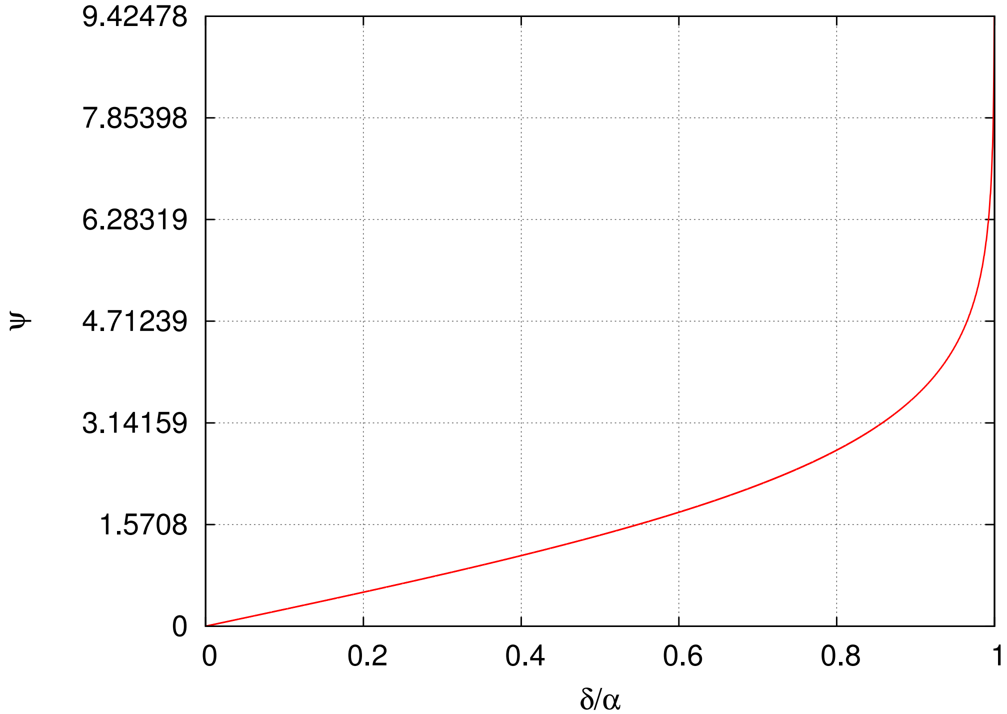

FIG. 2: The longitude change ψ as a function of δ/α.

```
print OUT qq|set xlabel "{/Symbol d}/{/Symbol a}"; set ylabel "{/Symbol y}"\n|;
print OUT qq|set xrange [0:1]\n|;
print OUT qq|set grid; unset key; set yrange [0:3*pi]; set ytics pi/2.\n|;
print OUT qq|set style line 1 lt 1 lw 2 pt 1 ps 1 lc rgb "#ff0000"\n|;
print OUT qq|plot "-" using 1:2 with lines ls 1\n|;
for($da=.0001;$da<1;$da+=.0002) {
  $integ = 0;
  $N = 50000;
  for $i (0..$N-1) {
    $w = ($i+.5)/$N;
    $integ += 1./sqrt( 1. - 27./4.*$da**2*$w**2*(1-$w) );
  }
  $integ *= 1.5*sqrt(3)*$da/$N;
  print OUT (sprintf "%7.5f %11.5E\n", $da, $integ);
}
print OUT "e\n";
close OUT;
```

(d) [4 points] *At what value of δ/α do you see the first ring that corresponds to the constellations at nadir ($\psi = \pi$)? And where is the first ring that corresponds to the constellations at zenith, but seen by a path that wraps around the black hole ($\psi = 2\pi$)?*

Inspection of the output file (`datadump-7.cdat` in the above script) shows that:

- One sees the nadir ($\psi = \pi$) at $\delta/\alpha = 0.858$ (i.e., 85.8% of the way from the center to the edge).
- One sees the zenith again ($\psi = 2\pi$) at $\delta/\alpha = 0.992$ (i.e., 99.2% of the way from the center to the edge).

So the multiple images of the sky get squeezed into a very narrow ring around the edge of the region "open to the sky."

(e) [4 points] *Draw, approximately to scale, a diagram of what $\mathcal{B}$ sees when they look up.*

See Fig. 3.

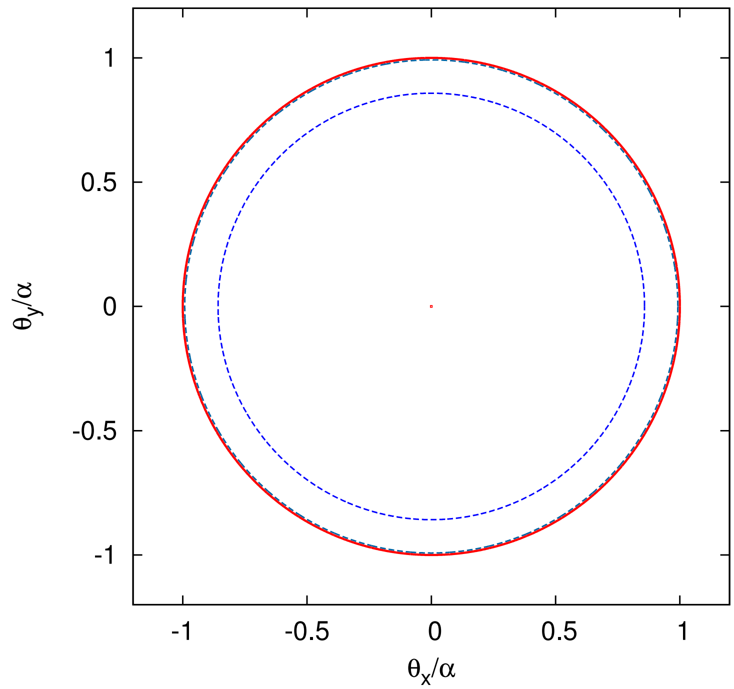

FIG. 3: The sky as seen from observer $\mathcal{B}$ looking up. The outer circle has radius α. The inner point is the zenith. The two dashed circles correspond to the first image of the nadir (inner circle, $\psi = \pi$) and the next image of the zenith (outer circle, $\psi = 2\pi$). In the exterior of the outer circle, observer $\mathcal{B}$ sees the black hole.