Physics 6820 – Homework 7

(Dated: Due: November 8, 2019)

1. The view from an observer hovering just above the horizon. [28 points]

In Lecture XVIII, §IIE, we considered observer $\mathcal{B}$ who hovers just above the horizon, remaining stationary with respect to the black hole, at $r = 2M(1+\epsilon)$, with $\epsilon \ll 1$. We showed that if observer $\mathcal{B}$ looks upward, at an angle δ from the zenith, they can see the distant sky if

$$\delta < \alpha \approx \frac{3\sqrt{3}}{2}\sqrt{\epsilon}. \tag{1}$$

(The region outside radius α appears to be covered up by the black hole.) In class, I told you that if $\mathcal{B}$ looks at an angle δ with $0 < \delta < \alpha$ from the vertical, they can see constellations that are in reality a distance ψ from the zenith. Clearly $\psi = 0$ if $\delta = 0$, and as δ increases one reaches $\psi = \pi$ (the image of the constellations at nadir, but turned into a ring), then $\psi = 2\pi$ (the image of constellations at zenith, but also turned into a ring), etc. The objective of this problem is to determine the mapping between ψ and δ.

You should give your answers to this problem to lowest order in δ and ϵ.

(a) [4 points] Assuming $\delta \ll 1$, show that the ratio $\tilde{\mathcal{L}}/\tilde{\mathcal{E}}$ of angular momentum to energy for the incoming photon is

$$\frac{\tilde{\mathcal{L}}}{\tilde{\mathcal{E}}} = 3\sqrt{3}\,M\frac{\delta}{\alpha}. \tag{2}$$

(b) [6 points] Show that for a photon on an equatorial orbit (placing $\mathcal{B}$ over the equator for simplicity):

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = \pm\frac{3\sqrt{3}\,M(\delta/\alpha)}{r^2\sqrt{1-27(\delta/\alpha)^2(M/r)^2(1-2M/r)}} \tag{3}$$

(c) [10 points] Express ψ in terms of an integral over part (b), and make a plot of ψ vs. δ/α. Your result should not depend explicitly on ϵ as long as ϵ is very small. (You may want to do the integral numerically.)

(d) [4 points] At what value of δ/α do you see the first ring that corresponds to the constellations at nadir ($\psi = \pi$)? And where is the first ring that corresponds to the constellations at zenith, but seen by a path that wraps around the black hole ($\psi = 2\pi$)?

(e) [4 points] Draw, approximately to scale, a diagram of what $\mathcal{B}$ sees when they look up.