Physics 6820 – Homework 6 Solutions

1. Gravitational wave detection by pulsar timing. [30 points]

Gravitational waves with periods of the order of years are expected to be emitted by the supermassive black holes in the centers of galaxies when they merge. These periods are too long for laser interferometers such as LIGO, or even planned space-based interferometers with "arms" millions of kilometers long (LISA). However, there are searches for these gravitational waves with pulsar timing.

For the sake of this problem, let's imagine a plane gravitational wave traveling through our galaxy in the $+z$ *direction and with the* $+$ *polarization:* $h_+ = A\cos\Omega(t-z-t_0)$, $h_\times = 0$, *where* A *is the strain amplitude and* t_0 *is the arrival time of a crest at Earth. We will place Earth at the origin and at rest in the coordinate frame in TT gauge, and place a pulsar at some distance* L *from the Earth at spatial coordinates* $(L\sin\theta_\star\cos\phi_\star, L\sin\theta_\star\sin\phi_\star, L\cos\theta_\star)$. *We will allow for any relation of* L *to* Ω, *i.e., in this problem do not assume anything about the numerical value of* ΩL *until the end.*

The pulsar is rapidly rotating and through a still-unknown mechanism sends pulses of radio waves, one after each rotation period P. *Each pulse travels toward the Earth, where it is received and assigned a time stamp. To a first approximation, these pulses arrive one after another, separated by time* P, *but there is a residual* $\delta t_{\rm arr}$ *created by the gravitational wave. The objective of this problem is to determine* $\delta t_{\rm arr}$ *in the idealized situation described above. You may work the entire problem to first order in* A *(or in the metric perturbation).*

(a) [4 points] *First take the unperturbed Minkowski metric. Show that the signal arriving at Earth at time* $t_{\rm arr}$ *left the pulsar at time* $t_{\rm arr} - L$, *and describe the path* $x^i(t)$ *taken by the radio pulse.*

The photon's 3-velocity has unit length and points in the opposite direction to the direction to the pulsar, so

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = (-\sin\theta_\star\cos\phi_\star, -\sin\theta_\star\sin\phi_\star, -\cos\theta_\star). \tag{1}$$

The photon is at the spatial origin at $t = t_{\rm arr}$, so the solution is

$$x^i(t) = ((t_{\rm arr}-t)\sin\theta_\star\cos\phi_\star, (t_{\rm arr}-t)\sin\theta_\star\sin\phi_\star, (t_{\rm arr}-t)\cos\theta_\star). \tag{2}$$

Inspection then shows that $x^i(t)$ is equal to the spatial coordinates of the pulsar at $t_{\rm arr} - t = L$ or $t = t_{\rm arr} - L$.

(b) [3 points] *Consider the path of a photon,* $x^\alpha(\sigma)$, *that is a null geodesic in the background Minkowski space, starting from* $\sigma_{\rm start}$ *and ending at* $\sigma_{\rm end}$. *Now suppose that in the perturbed metric* $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, *the photon takes a perturbed path* $x^\alpha(\sigma) + \delta x^\alpha(\sigma)$. *Show that*

$$0 = h_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} + 2\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}\delta x^\nu}{\mathrm{d}\sigma}. \tag{3}$$

Since photons travel along a null path, we have

$$0 = g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} \tag{4}$$

(where $g_{\mu\nu}$ is a function of position). If we do a linear perturbation of this, we find in general

$$0 = (\delta g_{\mu\nu} + g_{\mu\nu,\alpha}\delta x^\alpha)\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} + g_{\mu\nu}\frac{\mathrm{d}\delta x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} + g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}\delta x^\nu}{\mathrm{d}\sigma}. \tag{5}$$

In our case, the background $g_{\mu\nu}$ is the Minkowski metric $\eta_{\mu\nu}$, and the background partial derivative $g_{\mu\nu,\alpha}\delta x^\alpha = 0$. The perturbation $\delta g_{\mu\nu} = h_{\mu\nu}$, and by symmetry of $g_{\mu\nu}$ the last two terms are equal. This result then simplifies to Eq. (3).

(c) [9 points] *Now suppose that we consider the case of part (b), using* $\sigma = t$ *on the unperturbed path, and then integrate Eq. (3) over* σ *from the start to the end point. Show that the integral of the second term reduces to a boundary term in terms of displacements* δx^α *at the stars and end points. If we assume no perturbation to the start point* $\delta x^\alpha(\text{start}) = 0$, *and that at the end point only the arrival time is perturbed,* $\delta t(\text{end}) = \delta t_{\rm arr}$ *but* $\delta x^i(\text{end}) = 0$ *(because "arrival" is always measured at Earth), then*

$$\delta t_{\rm arr} = \frac{1}{2}\int_{t_{\rm arr}-L}^{t_{\rm arr}} h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\,\mathrm{d}t. \tag{6}$$

Plugging $\sigma = t$ into the terms with the unperturbed path (and the unperturbed path only – not the variation!) in Eq. (3) gives

$$0 = h_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + 2\eta_{\mu\nu}\dot{x}^\mu \frac{\mathrm{d}\delta x^\nu}{\mathrm{d}\sigma}. \tag{7}$$

Now we may simplify this by noting in the first term that $h_{\mu\nu}$ has only spatial parts; thus we write $h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = h_{ij}\dot{x}^i\dot{x}^j$. In the second term, we use that $\dot{x}^\mu =$constant. We do the integral $\mathrm{d}\sigma$ from the start to the end and get:

$$0 = \int_{\text{start}}^{\text{end}} h_{ij}\dot{x}^i\dot{x}^j\,\mathrm{d}t + 2\eta_{\mu\nu}\dot{x}^\mu \int_{\text{start}}^{\text{end}} \frac{\mathrm{d}\delta x^\nu}{\mathrm{d}\sigma}\,\mathrm{d}\sigma. \tag{8}$$

Plugging in the $\dot{x}^\mu$ and noting that the integral evaluates to $\delta x^\nu|_{\text{start}}^{\text{end}}$ gives

$$0 = \int_{t_1-L}^{t_1} h_{ij}\dot{x}^i\dot{x}^j\,\mathrm{d}t + 2\left[-\delta t - \sin\theta_\star\cos\phi_\star\,\delta x^1 - \sin\theta_\star\sin\phi_\star\,\delta x^2 - \cos\theta_\star\,\delta x^3\right]\Big|_{\text{start}}^{\text{end}}. \tag{9}$$

Now $\delta x^i = 0$ at the start and the end because the start is at the pulsar and the end is at the Earth, and both have fixed spatial position. Moreover, $\delta t = 0$ at the start since there is no change in the mapping of proper to coordinate time at the pulsar, and the pulsar emits pulses at a regular interval in pulsar proper time. By construction $\delta t = \delta t_{\text{arr}}$ at the end point. Therefore:

$$0 = \int_{t_1-L}^{t_1} h_{ij}\dot{x}^i\dot{x}^j\,\mathrm{d}t + 2(-\delta t_{\text{arr}}). \tag{10}$$

The result – Eq. (6) – follows.

(d) [9 points] *Plug into your result from part (c) the specific form of the metric perturbation and do the integral to find that*

$$\delta t_{\text{arr}} = 2A\sin^2\theta_\star\cos 2\phi_\star \frac{\sin[\Omega(1+\cos\theta_\star)\frac{L}{2}]}{\Omega(1+\cos\theta_\star)} \cos\left[\Omega\left(t_{\text{arr}} - \frac{1+\cos\theta_\star}{2}L - t_0\right)\right]. \tag{11}$$

[You will have to substitute in the unperturbed trajectory to get $z(t)$, and at the end use the sum-to-product trigonometric identity.]

We see that

$$\begin{aligned} h_{ij}\dot{x}^i\dot{x}^j &= A\cos\Omega(t-z-t_0)\,[(\dot{x})^2 - (\dot{y})^2] \\ &= A\cos\Omega(t-z-t_0)\,[\sin^2\theta_\star\cos^2\phi_\star - \sin^2\theta_\star\sin^2\phi_\star] \\ &= A\cos\Omega(t-z-t_0)\sin^2\theta_\star\cos 2\phi_\star. \end{aligned} \tag{12}$$

Then

$$\begin{aligned} \delta t_{\text{arr}} &= A\sin^2\theta_\star\cos 2\phi_\star \int_{t_{\text{arr}}-L}^{t_{\text{arr}}} \cos\Omega(t-z-t_0)\,\mathrm{d}t \\ &= A\sin^2\theta_\star\cos 2\phi_\star \int_{t_{\text{arr}}-L}^{t_{\text{arr}}} \cos\Omega((1+\cos\theta_\star)t - \cos\theta_\star t_{\text{arr}} - t_0)\,\mathrm{d}t \\ &= A\sin^2\theta_\star\cos 2\phi_\star \frac{\sin\Omega[(1+\cos\theta_\star)t_{\text{arr}} - \cos\theta_\star t_{\text{arr}} - t_0] - \sin\Omega[(1+\cos\theta_\star)(t_{\text{arr}}-L) - \cos\theta_\star t_{\text{arr}} - t_0]}{\Omega(1+\cos\theta_\star)} \\ &= 2A\sin^2\theta_\star\cos 2\phi_\star \frac{\sin[\Omega(1+\cos\theta_\star)\frac{L}{2}]}{\Omega(1+\cos\theta_\star)} \cos\left[\Omega\left(t_{\text{arr}} - \frac{1+\cos\theta_\star}{2}L - t_0\right)\right]. \end{aligned} \tag{13}$$

In the next part of this problem, we will consider the semi-amplitude *of δt_{arr}, i.e., half of the maximum minus the minimum:*

$$\text{SemiAmp}[\delta t_{\text{arr}}] = \left| 2A\sin^2\theta_\star\cos 2\phi_\star \frac{\sin[\Omega(1+\cos\theta_\star)\frac{L}{2}]}{\Omega(1+\cos\theta_\star)} \right|. \tag{14}$$

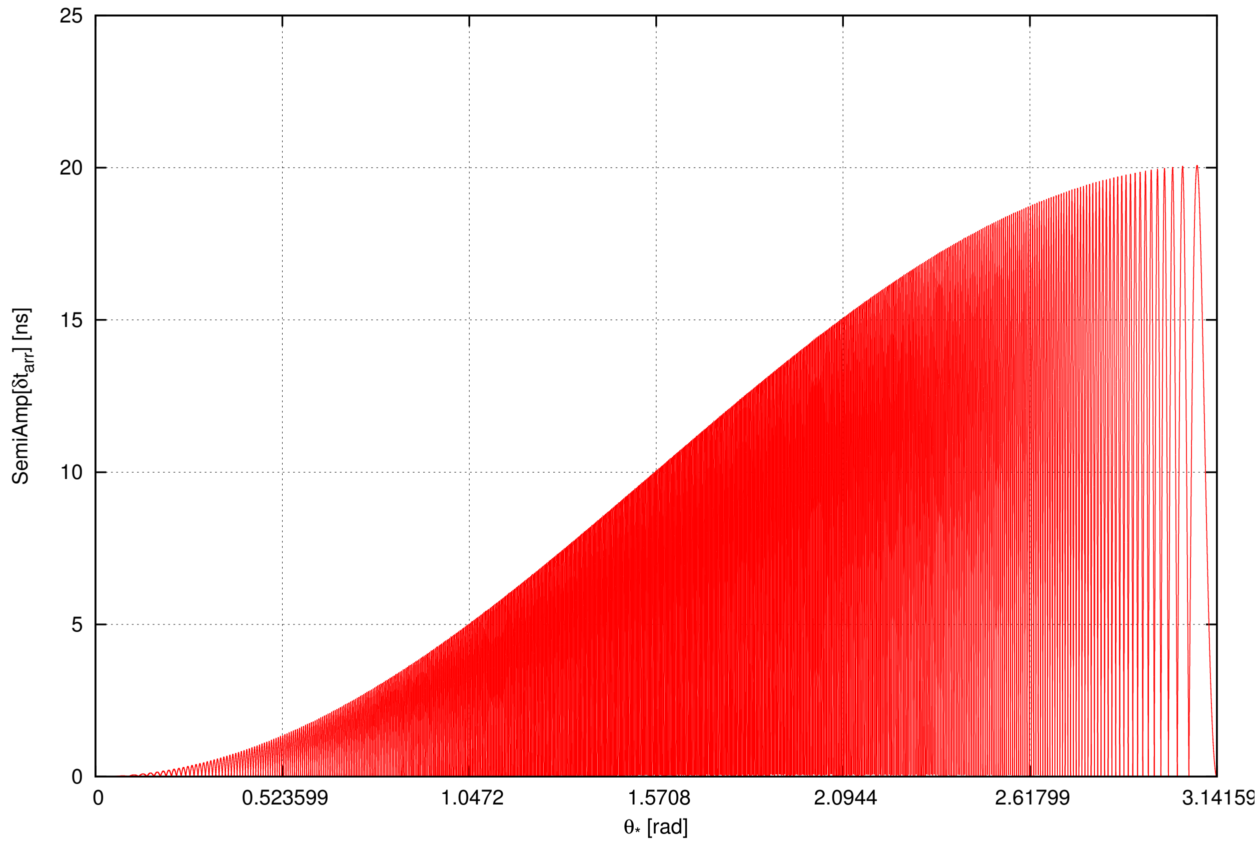

FIG. 1: Solution to part (e).

(e) [5 points] *For a pulsar at $L = 100$ pc, a period of $2\pi/\Omega = 1$ yr, and a strain of $A = 10^{-15}$, make a plot of SemiAmp$[\delta t_{\rm arr}]$ as a function of $\theta_\star$ for $0 \le \theta_\star \le \pi$, assuming zero longitude ($\phi_\star = 0$). Plot the vertical axis in an SI unit with prefix (s, ms, μs, ns, ps ... as you see fit). You might need a lot of resolution on the x-axis!*

Here is a gnuplot script:

```
set term postscript enhanced 12 eps color
set output "hw6ans.eps"
set xrange [0:pi]
A = 1e-15
Omega = 2*pi/365.25/86400. # converted from yr^-1 to s^-1
L = 100.*3.086e16/2.998e8 # in s; note divided by c = 2.998e8 m/s, and 1 pc = 3.086e16 m
set style line 1 lt 1 lw 1 pt 1 ps 1 lc 1
set samples 262144
set xlabel "{/Symbol q}_* [rad]"
set xtics pi/6.
set ylabel "SemiAmp[{/Symbol d}t_{arr}] [ns]"
unset key
set grid
plot 2*A*sin(x)**2*abs(sin(Omega*L/2.*(1+cos(x))))/Omega/(1+cos(x))/1e-9 ls 1
```

The output is shown in Fig. 1.