Physics 6820 – Homework 5 Solutions

General comment: – *Both of these problems are based on the results from HW 4. I haven't returned your HW 4 yet, so you can start from the results in the posted HW 4 Solution set.*

1. Geodesics on the sphere. [26 points]

This problem follows the method of solution to the geodesic equation that we did in class in polar coordinates.

(a) [4 points] *Write the geodesic equations on the sphere in the form $\ddot{\theta} =?$ and $\ddot{\phi} =?$, where the right-hand side is expressed in terms of θ, ϕ, $\dot{\theta}$, and $\dot{\phi}$, and the dot indicates a derivative with respect to arc length.*

Recall from HW #4 that the non-zero Christoffel symbols were

$$\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta, \quad \text{other 5 are zero,} \tag{1}$$

and that the geodesic equation is $\ddot{x}^\mu = -\Gamma^\mu{}_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$. Writing down the non-zero contributions, we get

$$\ddot{\theta} = \sin\theta\cos\theta\,\dot{\phi}^2, \quad \ddot{\phi} = -2\cot\theta\,\dot{\theta}\dot{\phi}, \tag{2}$$

where only $\Gamma^{\theta}{}_{\phi\phi}$ contributes in the first equation and $\Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta}$ in the second (having both Christoffel symbols gives the factor of 2).

(b) [5 points] *Show from these equations that the quantities*

$$\tilde{L} = R^2\sin^2\theta\,\dot{\phi} \quad \text{and} \quad \boldsymbol{u}\cdot\boldsymbol{u} = R^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) \tag{3}$$

are conserved. What is the interpretation of $\tilde{L}$?

Let us check the conservation by brute force comparison to Eq. (2). For $\tilde{L}$:

$$\frac{d\tilde{L}}{ds} = \frac{d}{ds}(R^2\sin^2\theta\,\dot{\phi}) = 2R^2\sin\theta\cos\theta\,\dot{\theta}\dot{\phi} + R^2\sin^2\theta\,\ddot{\phi} = 2R^2\sin\theta\cos\theta\,\dot{\theta}\dot{\phi} + R^2\sin^2\theta(-2\cot\theta\,\dot{\theta}\dot{\phi}) = 0. \tag{4}$$

For $\boldsymbol{u}\cdot\boldsymbol{u}$:

$$\begin{aligned}
\frac{d}{ds}(\boldsymbol{u}\cdot\boldsymbol{u}) &= R^2\frac{d}{ds}(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) \\
&= R^2\left[2\dot{\theta}\ddot{\theta} + 2\sin\theta\cos\theta\,\dot{\theta}\dot{\phi}^2 + 2\sin^2\theta\,\dot{\phi}\ddot{\phi}\right] \\
&= R^2\left[2\dot{\theta}(\sin\theta\cos\theta\,\dot{\phi}^2) + 2\sin\theta\cos\theta\,\dot{\theta}\dot{\phi}^2 + 2\sin^2\theta\,\dot{\phi}(-2\cot\theta\,\dot{\theta}\dot{\phi})\right] \\
&= R^2(2+2-4)\sin\theta\cos\theta\,\dot{\theta}\dot{\phi}^2 = 0.
\end{aligned} \tag{5}$$

Thus both of these quantities are indeed conserved.

Now we interpret $\tilde{L}$. Recall that $g_{\phi\phi} = R^2\sin^2\theta$ and $g_{\phi\theta} = 0$; thus we can see that $u_\phi = R^2\sin^2\theta\,u^\phi = R^2\sin^2\theta\,\dot{\phi} = \tilde{L}$. Thus $\tilde{L}$ is the conjugate momentum to longitude ϕ, i.e., the angular momentum around the z-axis.

In what follows, we will take the normalization from class that $\boldsymbol{u}\cdot\boldsymbol{u} = 1$.

(c) [5 points] *Use the result of (b) to write an equation for $\dot{\theta}$ in terms of θ and $\tilde{L}$. Then use this to show that s can be written as an integral over θ:*

$$s = \pm R\int\frac{d\theta}{\sqrt{1 - \frac{\tilde{L}^2}{R^2\sin^2\theta}}}. \tag{6}$$

Using the result that $R^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) = 1$, we see that

$$\dot{\theta} = \pm\sqrt{\frac{1}{R^2} - \sin^2\theta\,\dot{\phi}^2} = \pm\sqrt{\frac{1}{R^2} - \sin^2\theta\left(\frac{\tilde{L}}{R^2\sin^2\theta}\right)^2} = \pm\frac{1}{R}\sqrt{1 - \frac{\tilde{L}^2}{R^2\sin^2\theta}}. \tag{7}$$

Then since $\dot{\theta} = \mathrm{d}\theta/\mathrm{d}s$, we find:

$$s = \int \frac{\mathrm{d}\theta}{\dot{\theta}} = \pm R \int \frac{\mathrm{d}\theta}{\sqrt{1 - \frac{\tilde{L}^2}{R^2 \sin^2 \theta}}}. \tag{8}$$

(d) [6 points] *Evaluate this integral and show that*

$$\cos\theta = \sin I \cos\frac{s - C}{R}, \tag{9}$$

where we defined I by $\tilde{L} = R\cos I$. [Hint: the integral simplifies if you do the substitution $z = R\cos\theta$ and then $z = R\sin I \cos\psi$.]

Using the substitutions in the hint, and the trigonometric indentities:

$$\begin{aligned}
s &= \pm R \int \frac{\mathrm{d}\theta}{\sqrt{1 - \frac{\tilde{L}^2}{R^2 \sin^2 \theta}}} \\
&= \pm R \int \frac{-\mathrm{d}z/\sqrt{R^2 - z^2}}{\sqrt{1 - \frac{\tilde{L}^2}{R^2 - z^2}}} \\
&= \pm R \int \frac{-\mathrm{d}z}{\sqrt{R^2 - z^2 - \tilde{L}^2}} \\
&= \pm R \int \frac{-\mathrm{d}z}{\sqrt{R^2 \sin^2 I - z^2}} \\
&= \pm R \int \frac{R \sin I \sin\psi \, \mathrm{d}\psi}{\sqrt{R^2 \sin^2 I - R^2 \sin^2 I \cos^2 \psi}} \\
&= \pm R \int \mathrm{d}\psi.
\end{aligned} \tag{10}$$

This means $\psi = \pm(s - C)/R$, where C is the constant of integration. We thus have

$$\cos\theta = \frac{z}{R} = \sin I \cos\psi = \sin I \cos\frac{s - C}{R}, \tag{11}$$

where the $\pm$ no longer matters since the cosine function is even.

(e) [4 points] *Explain, based on your 3D Euclidean intuition, why a particle moving at constant speed along a great circle on a sphere should exhibit sinusoidal motion on the z-axis. What are the interpretations of z, I, and ψ in this picture?*

In uniform circular motion in 2D, the coordinates are usually described as $x = R\cos(s/R)$ and $y = R\sin(s/R)$, where s is path length. Here the circle is tilted, but after a linear coordinate transformation the coordinates of a particle should be linear combinations of $\cos(s/R)$ and $\sin(s/R)$. Of course, any such linear combination can be written in terms of $\cos[(s - C)/R]$ for some C using the sum-of-angles formula:

$$\cos\frac{s - C}{R} = \cos\frac{C}{R}\cos\frac{s}{R} + \sin\frac{C}{R}\sin\frac{s}{R} \tag{12}$$

(that is, we can change the phase so as to have only the cosine term).

The 3D physical interpretation of $z = R\cos\theta$ is of course that it is the z-coordinate of the particle. Here I is the inclination of the great circle (the latitude $\frac{\pi}{2} - \theta$ varies between $\pm I$). If we describe in 3D O as the center of the sphere, $\mathcal{A}$ as the point of most northerly latitude, and $\mathcal{P}$ as the point on the particle's path, then ψ can be interpreted as the 3D angle $\angle \mathcal{A}O\mathcal{P}$, because the particle is at $\mathcal{A}$ at $s = C$, and then at general s the arc length from $\mathcal{A}$ to $\mathcal{P}$ is $s - C$ – hence the angle is this divided by R.

(f) [2 points] *Now show that*

$$\phi = \int \frac{\tilde{L}}{R^2 \sin^2 \theta} \, \mathrm{d}s, \tag{13}$$

where the function $\theta(s)$ is given in part (d).

This is because $\mathrm{d}\phi/\mathrm{d}s = \dot\phi = \tilde L/(R^2\sin^2\theta)$; then we may integrate.

Comment — In the interest of keeping this problem at reasonable length, I won't ask you to go through the mechanics of the ϕ integral. It can be solved by trigonometric substitution, which shouldn't surprise you given that it is associated with great circles on a sphere. The answer turns out to be

$$\phi = \tan^{-1}\left(\sec I \tan\frac{s-C}{R}\right) + \phi_0, \tag{14}$$

where ϕ_0 is the integration constant.

The problem didn't ask to do the integral of Eq. (13), but I'll go ahead and do it here in case you want to see how it is done.

$$\begin{aligned}
\phi &= \int \frac{\tilde L}{R^2\sin^2\theta}\,\mathrm{d}s \\
&= \frac{\cos I}{R}\int \frac{1}{\sin^2\theta}\,\mathrm{d}s \\
&= \frac{\cos I}{R}\int \frac{1}{1-(\sin I\cos\frac{s-C}{R})^2}\,\mathrm{d}s \\
&= \cos I\int \frac{1}{1-\sin^2 I\cos^2\psi}\,\mathrm{d}\psi.
\end{aligned} \tag{15}$$

We now use the substitution $\xi = \tan\psi$, so $\cos^2\psi = 1/(1+\xi^2)$ and $\mathrm{d}\psi = \mathrm{d}\xi/(1+\xi^2)$:

$$\begin{aligned}
\phi &= \cos I\int \frac{1}{1-(\sin^2 I)/(1+\xi^2)}\,\frac{\mathrm{d}\xi}{1+\xi^2} \\
&= \cos I\int \frac{1}{1+\xi^2-\sin^2 I}\,\mathrm{d}\xi \\
&= \cos I\int \frac{1}{\cos^2 I+\xi^2}\,\mathrm{d}\xi.
\end{aligned} \tag{16}$$

This suggests the further substitution $\xi = (\cos I)\alpha$:

$$\begin{aligned}
\phi &= \cos I\int \frac{1}{\cos^2 I+(\cos^2 I)\alpha^2}\cos I\,\mathrm{d}\alpha \\
&= \int \frac{1}{1+\alpha^2}\,\mathrm{d}\alpha \\
&= \tan^{-1}\alpha + \phi_0 \\
&= \tan^{-1}\left(\sec I\tan\frac{s-C}{R}\right) + \phi_0.
\end{aligned} \tag{17}$$

2. Curvature of the sphere. [12 points]

Consider the 2-dimensional surface of the sphere from Homework #4. We will consider the Riemann curvature tensor on the sphere.

(a) [5 points] *Show that*

$$R^\theta{}_{\phi\theta\phi} = \sin^2\theta. \tag{18}$$

We compute:

$$\begin{aligned}
R^\theta{}_{\phi\theta\phi} &= \Gamma^\theta{}_{\phi\phi,\theta} + \Gamma^\theta{}_{\delta\theta}\Gamma^\delta{}_{\phi\phi} - \Gamma^\theta{}_{\phi\theta,\phi} - \Gamma^\theta{}_{\delta\phi}\Gamma^\delta{}_{\phi\theta} \\
&= \Gamma^\theta{}_{\phi\phi,\theta} - \Gamma^\theta{}_{\phi\phi}\Gamma^\phi{}_{\phi\theta} \\
&= (-\sin\theta\cos\theta)_{,\theta} - (-\sin\theta\cos\theta)\cot\theta \\
&= -\cos^2\theta + \sin^2\theta + \cos^2\theta \\
&= \sin^2\theta.
\end{aligned} \tag{19}$$

In the second line, we included only the specific terms that are non-zero.

(b) [4 points] *Define an orthonormal basis* $\{\boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\phi}}\}$ *with basis vectors parallel to the usual coordinate axes. Show that*

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{1}{R^2}. \tag{20}$$

Since $g_{\theta\phi} = 0$, the vectors $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\phi$ are already orthogonal. We can write normalized versions of these vectors as:

$$\boldsymbol{e}_{\hat{\theta}} = \frac{1}{\sqrt{g_{\theta\theta}}}\boldsymbol{e}_\theta = \frac{1}{R}\boldsymbol{e}_\theta \tag{21}$$

and

$$\boldsymbol{e}_{\hat{\phi}} = \frac{1}{\sqrt{g_{\phi\phi}}}\boldsymbol{e}_\phi = \frac{1}{R\sin\theta}\boldsymbol{e}_\phi. \tag{22}$$

To express the Riemann tensor, let us first lower the θ index on Eq. (18) to get:

$$R_{\theta\phi\theta\phi} = g_{\theta\theta}R^{\theta}{}_{\phi\theta\phi} = R^2\sin^2\theta. \tag{23}$$

Finally, doing the basis change:

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{1}{R}\frac{1}{R\sin\theta}\frac{1}{R}\frac{1}{R\sin\theta}R_{\theta\phi\theta\phi} = \frac{1}{R}\frac{1}{R\sin\theta}\frac{1}{R}\frac{1}{R\sin\theta}R^2\sin^2\theta = \frac{1}{R^2}. \tag{24}$$

(c) [3 points] *Use the symmetries described in class to write the remaining components of the Riemann tensor (in 2 dimensions, it has only 1 independent component).*

The antisymmetry properties of the Riemann tensor guarantee that if the first two *or* the last two indices are the same, then that component is zero. So we write:

$$R_{\hat{\theta}\hat{\theta}\hat{\theta}\hat{\phi}} = R_{\hat{\phi}\hat{\phi}\hat{\theta}\hat{\phi}} = R_{\hat{\theta}\hat{\theta}\hat{\phi}\hat{\theta}} = R_{\hat{\phi}\hat{\phi}\hat{\phi}\hat{\theta}} = R_{\hat{\theta}\hat{\theta}\hat{\theta}\hat{\theta}} = R_{\hat{\phi}\hat{\phi}\hat{\theta}\hat{\theta}} = R_{\hat{\theta}\hat{\theta}\hat{\phi}\hat{\phi}} = R_{\hat{\phi}\hat{\phi}\hat{\phi}\hat{\phi}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\theta}} = R_{\hat{\phi}\hat{\theta}\hat{\theta}\hat{\theta}} = R_{\hat{\theta}\hat{\phi}\hat{\phi}\hat{\phi}} = R_{\hat{\phi}\hat{\theta}\hat{\phi}\hat{\phi}} = 0. \tag{25}$$

There are 4 remaining components, and they can be derived from Eq. (24):

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = R_{\hat{\phi}\hat{\theta}\hat{\phi}\hat{\theta}} = \frac{1}{R^2} \quad \text{and} \quad R_{\hat{\phi}\hat{\theta}\hat{\theta}\hat{\phi}} = R_{\hat{\theta}\hat{\phi}\hat{\phi}\hat{\theta}} = -\frac{1}{R^2}. \tag{26}$$

Comment — The curvature tensor expressed in any local orthonormal basis is independent of position on the sphere (as it should be) and goes to zero as the radius of the sphere goes to ∞ *(as it should).*