Physics 6820 – Homework 5

(Dated: Due: October 4, 2019)

General comment: – Both of these problems are based on the results from HW 4. I haven't returned your HW 4 yet, so you can start from the results in the posted HW 4 Solution set.

1. Geodesics on the sphere. [26 points]

This problem follows the method of solution to the geodesic equation that we did in class in polar coordinates.

(a) [4 points] Write the geodesic equations on the sphere in the form $\ddot{\theta} =?$ and $\ddot{\phi} =?$, where the right-hand side is expressed in terms of θ, ϕ, $\dot{\theta}$, and $\dot{\phi}$, and the dot indicates a derivative with respect to arc length.

(b) [5 points] Show from these equations that the quantities

$$\tilde{L} = R^2 \sin^2\theta\,\dot{\phi} \quad \text{and} \quad \boldsymbol{u}\cdot\boldsymbol{u} = R^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) \tag{1}$$

are conserved. What is the interpretation of $\tilde{L}$?

In what follows, we will take the normalization from class that $\boldsymbol{u}\cdot\boldsymbol{u} = 1$.

(c) [5 points] Use the result of (b) to write an equation for $\dot{\theta}$ in terms of θ and $\tilde{L}$. Then use this to show that s can be written as an integral over θ:

$$s = \pm R \int \frac{\mathrm{d}\theta}{\sqrt{1 - \frac{\tilde{L}^2}{R^2\sin^2\theta}}}. \tag{2}$$

(d) [6 points] Evaluate this integral and show that

$$\cos\theta = \sin I \cos\frac{s-C}{R}, \tag{3}$$

where we defined I by $\tilde{L} = R\cos I$. [Hint: the integral simplifies if you do the substitution $z = R\cos\theta$ and then $z = R\sin I\cos\psi$.]

(e) [4 points] Explain, based on your 3D Euclidean intuition, why a particle moving at constant speed along a great circle on a sphere should exhibit sinusoidal motion on the z-axis. What are the interpretations of z, I, and ψ in this picture?

(f) [2 points] Now show that

$$\phi = \int \frac{\tilde{L}}{R^2\sin^2\theta}\,\mathrm{d}s, \tag{4}$$

where the function $\theta(s)$ is given in part (d).

Comment — In the interest of keeping this problem at reasonable length, I won't ask you to go through the mechanics of the ϕ integral. It can be solved by trigonometric substitution, which shouldn't surprise you given that it is associated with great circles on a sphere. The answer turns out to be

$$\phi = \tan^{-1}\left(\sec I \tan\frac{s-C}{R}\right) + \phi_0, \tag{5}$$

where ϕ_0 is the integration constant.

2. Curvature of the sphere. [12 points]

Consider the 2-dimensional surface of the sphere from Homework #4. We will consider the Riemann curvature tensor on the sphere.

(a) [5 points] Show that

$$R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta. \tag{6}$$

(b) [4 points] Define an orthonormal basis $\{\boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\phi}}\}$ with basis vectors parallel to the usual coordinate axes. Show that

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{1}{R^2}. \tag{7}$$

(c) [3 points] Use the symmetries described in class to write the remaining components of the Riemann tensor (in 2 dimensions, it has only 1 independent component).

Comment — The curvature tensor expressed in any local orthonormal basis is independent of position on the sphere (as it should be) and goes to zero as the radius of the sphere goes to ∞ (as it should).