Physics 6820 – Homework 4 Solutions

1. Practice with Christoffel symbols. [24 points]

This problem considers the geometry of a 2-sphere of radius R. We build the usual coordinate system on the sphere, with coordinates θ (colatitude, i.e., $\frac{\pi}{2}$ minus the latitude) and ϕ (longitude).

(a) [4 points] *Show that the line element is*

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2. \tag{1}$$

(I am looking for a geometric argument: you should show a diagram, a few right triangles, and some simple reasoning with trigonometry and the Pythagorean theorem. However I will accept any valid alternative solution.)

A diagram showing two points $\mathcal{P}$ and $\mathcal{Q}$, infintesimally separated in colatitude and longitude, is shown in Fig. 1. The length of $\mathcal{PB}$ is $R\,d\theta$, and that of $\mathcal{BQ}$ is $R\sin\theta\,d\phi$ since the circle at colatitude θ has radius $|\mathcal{AP}| = R\sin\theta$. We conclude by the Pythagorean theorem that

$$ds^2 = |\mathcal{PB}|^2 + |\mathcal{BQ}|^2 = R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2. \tag{2}$$

(b) [12 points] *Find the metric tensor components $g_{\mu\nu}$ and the Christoffel symbols $\Gamma^\mu{}_{\alpha\beta}$ on the sphere.*

The metric tensor components resulting from the line element in part (a) are (reading off from $ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu$)

$$g_{\theta\theta} = R^2, \quad g_{\phi\phi} = R^2\sin^2\theta \quad g_{\theta\phi} = g_{\phi\theta} = 0. \tag{3}$$

We see that the only non-zero metric component derivative is:

$$g_{\phi\phi,\theta} = 2R^2\sin\theta\cos\theta, \quad \text{other 7 are zero.} \tag{4}$$

This implies that the non-zero Christoffel symbols are

$$\Gamma_{\theta\phi\phi} = -R^2\sin\theta\cos\theta, \quad \Gamma_{\phi\theta\phi} = \Gamma_{\phi\phi\theta} = R^2\sin\theta\cos\theta, \quad \text{other 5 are zero.} \tag{5}$$

We can now use the inverse metric:

$$g_{\mu\nu} \to \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}, \quad g^{\mu\nu} \to \begin{pmatrix} R^{-2} & 0 \\ 0 & R^{-2}\csc^2\theta \end{pmatrix} \tag{6}$$

to raise the first indices on the Christoffel symbols. We get:

$$\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta, \quad \text{other 5 are zero} \tag{7}$$

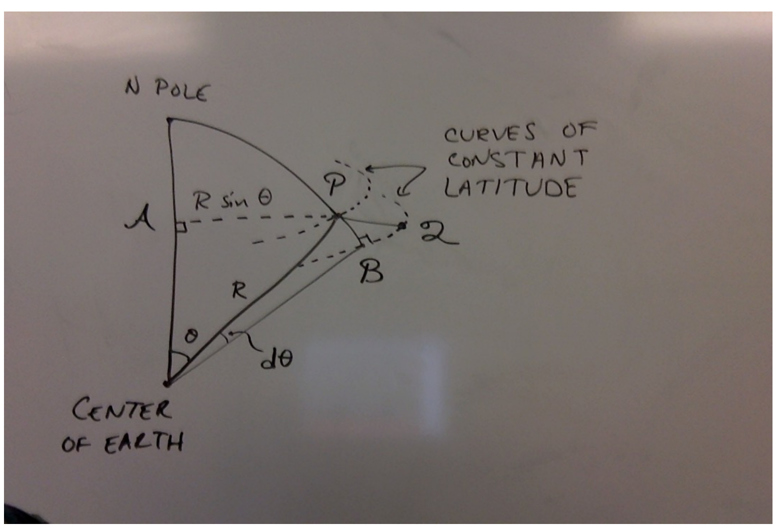

FIG. 1: Diagram for part (a).

(for the latter symbol, we had to use $\csc^2\theta \sin\theta\cos\theta = \cot\theta$).

(c) [8 points] *Take a map of the continental United States with lines of latitude and longitude drawn, and showing the lines of latitude curved (e.g., in Lambert, Albers, or similar projection; not Mercator). Draw on this map the vector fields* $\boldsymbol{e}_\theta$ *and* $\boldsymbol{e}_\phi$*. Explain using the diagrams of these fields the geometric significance of the non-zero Christoffel symbols.*

[You will probably want to download and print a map from the Internet; if so please provide a reference. If it helps, you can print it in black and white and draw the vectors in color, but any reasonably clear indication of the two vector fields is fine. Note that I chose the continental US because it is a region large enough to see the curvature of the latitude-longitude coordinate system, but not so large that any map projection would be hopelessly distorted. The issues associated with the latter will be covered next week, when we study curvature and see that it involves the second derivative of the metric tensor.]

Fig. 2 is a US Government (public domain) map of the US; I have overplotted $\boldsymbol{e}_\theta$ (orange dashed arrows, pointed South) and $\boldsymbol{e}_\phi$ (blue solid arrows, pointed East). Note that the vectors are the velocity vectors of points with $\dot\theta = 1, \dot\phi = 0$ and $\dot\theta = 0, \dot\phi = 1$, so the "length" of the $\boldsymbol{e}_\phi$ vectors increases as we move toward the Equator. In the Northern Hemisphere, $0 < \theta < \frac{\pi}{2}$ so $\sin\theta$ and $\cos\theta$ are both positive.

The Christoffel symbols tell us that:

$$\nabla_{\boldsymbol{e}_\theta}\boldsymbol{e}_\theta = 0, \quad \nabla_{\boldsymbol{e}_\theta}\boldsymbol{e}_\phi = \cot\theta\,\boldsymbol{e}_\phi, \quad \nabla_{\boldsymbol{e}_\phi}\boldsymbol{e}_\theta = \cot\theta\,\boldsymbol{e}_\phi, \quad \text{and} \quad \nabla_{\boldsymbol{e}_\phi}\boldsymbol{e}_\phi = -\sin\theta\cos\theta\,\boldsymbol{e}_\theta. \tag{8}$$

These make sense:

- We have $\nabla_{\boldsymbol{e}_\theta}\boldsymbol{e}_\theta = 0$ because as one moves south, the $\boldsymbol{e}_\theta$ vector (south at constant length) does not change.
- We have $\nabla_{\boldsymbol{e}_\theta}\boldsymbol{e}_\phi = \cot\theta\,\boldsymbol{e}_\phi$ because as one moves south, the $\boldsymbol{e}_\phi$ vector gets longer (lines of longitude are farther apart).
- We have $\nabla_{\boldsymbol{e}_\phi}\boldsymbol{e}_\theta = \cot\theta\,\boldsymbol{e}_\phi$ because as one moves east, the southward-pointing vector rotates (and rotates toward the east).
- We have $\nabla_{\boldsymbol{e}_\phi}\boldsymbol{e}_\phi = -\sin\theta\cos\theta\,\boldsymbol{e}_\theta$ because as one moves east, the eastward-pointing vector rotates (and rotates toward the north, i.e., $-\boldsymbol{e}_\theta$).

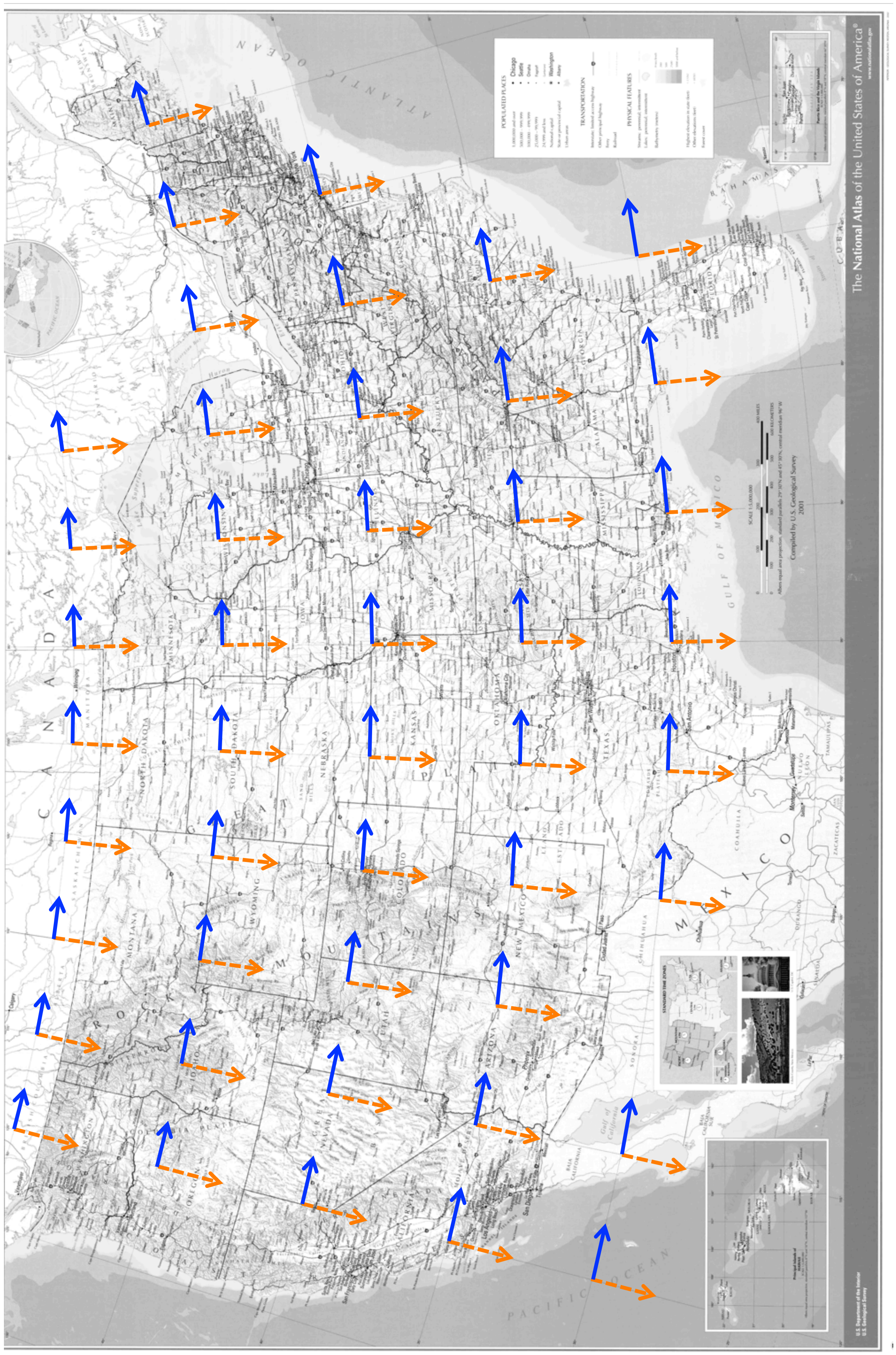

FIG. 2: Map of the US for part (c).