Physics 6820 – Homework 4

(Dated: Due: September 20, 2019)

1. Practice with Christoffel symbols. [24 points]

This problem considers the geometry of a 2-sphere of radius R. We build the usual coordinate system on the sphere, with coordinates θ (colatitude, i.e., $\frac{\pi}{2}$ minus the latitude) and ϕ (longitude).

(a) [4 points] Show that the line element is

$$\mathrm{d}s^2 = R^2 \mathrm{d}\theta^2 + R^2 \sin^2\theta \, \mathrm{d}\phi^2. \tag{1}$$

(I am looking for a geometric argument: you should show a diagram, a few right triangles, and some simple reasoning with trigonometry and the Pythagorean theorem. However I will accept any valid alternative solution.)

(b) [12 points] Find the metric tensor components $g_{\mu\nu}$ and the Christoffel symbols $\Gamma^{\mu}{}_{\alpha\beta}$ on the sphere.

(c) [8 points] Take a map of the continental United States with lines of latitude and longitude drawn, and showing the lines of latitude curved (e.g., in Lambert, Albers, or similar projection; not Mercator). Draw on this map the vector fields $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\phi$. Explain using the diagrams of these fields the geometric significance of the non-zero Christoffel symbols.

[You will probably want to download and print a map from the Internet; if so please provide a reference. If it helps, you can print it in black and white and draw the vectors in color, but any reasonably clear indication of the two vector fields is fine. Note that I chose the continental US because it is a region large enough to see the curvature of the latitude-longitude coordinate system, but not so large that any map projection would be hopelessly distorted. The issues associated with the latter will be covered next week, when we study curvature and see that it involves the second derivative of the metric tensor.]