Physics 6820 – Homework 3 Solutions

1. Maxwell's equations. [32 points]
In this problem, we consider the action for some particles and the electromagnetic field in special relativity:

$$S = \sum_{\text{particles}} \left[-m \int d\tau + q \int A_\mu(\boldsymbol{x})\, \mathrm{d}x^\mu \right] + \int -\frac{1}{4} F_{\gamma\mu} F^{\gamma\mu} \mathrm{d}^4\boldsymbol{x}. \tag{1}$$

Here $\boldsymbol{A}$ is a 1-form, and in the sum over particles, m is the mass of the particle and q is its electric charge. The 2-form electromagnetic field tensor is $\boldsymbol{F} = \tilde{\boldsymbol{d}}\boldsymbol{A}$. We have used units where $\epsilon_0 = 1$ (and since $c = 1$, we will also have $\mu_0 = 1$).

On Homework #2, you showed that the particle acceleration satisfied $ma_\gamma = qF_{\gamma\mu}u^\mu$, and that the field components were

$$F_{\gamma\mu} \to \left(\begin{array}{c|ccc} 0 & -E_x & -E_y & -E_z \\ \hline E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{array} \right). \tag{2}$$

(a) [4 points] *Show that the* $\sum_{\text{particles}} q \int A_\mu(\boldsymbol{x})\, \mathrm{d}x^\mu$ *term in the action can be written as*

$$\int A_\mu J^\mu \, \mathrm{d}^4\boldsymbol{x}, \tag{3}$$

where the 4-current density $\boldsymbol{J}$ *for a swarm of particles is:*

$$J^\mu(\boldsymbol{y}) = \sum_{\text{particles}} q \int \delta^{(4)}(\boldsymbol{x}(\sigma) - \boldsymbol{y}) \frac{\mathrm{d}x^\mu(\sigma)}{\mathrm{d}\sigma} \, \mathrm{d}\sigma. \tag{4}$$

We can re-express the interaction term by inserting an integral and a Dirac δ-function:

$$\begin{aligned} S_{\text{int}} &= \sum_{\text{particles}} q \int A_\mu(\boldsymbol{x})\, \mathrm{d}x^\mu \\ &= \sum_{\text{particles}} q \int A_\mu(\boldsymbol{x}) \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\, \mathrm{d}\sigma \\ &= \sum_{\text{particles}} q \int \int A_\mu(\boldsymbol{x}(\sigma)) \delta^{(4)}(\boldsymbol{x}(\sigma) - \boldsymbol{y}) \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\, \mathrm{d}\sigma\, \mathrm{d}^4\boldsymbol{y} \\ &= \int A_\mu(\boldsymbol{y}) \sum_{\text{particles}} q \int \delta^{(4)}(\boldsymbol{x}(\sigma) - \boldsymbol{y}) \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\, \mathrm{d}\sigma\, \mathrm{d}^4\boldsymbol{y} \\ &= \int A_\mu(\boldsymbol{y}) J^\mu(\boldsymbol{y})\, \mathrm{d}^4\boldsymbol{y}. \end{aligned} \tag{5}$$

(b) [3 points] *Explain (maybe with words and a few equations) why J^0 is the usual charge density and J^i are the components of the usual current density.*

The current density is parameterization-independent (i.e., we can choose any coordinate along the line to be σ), so let's take $\sigma = t$. In this case, Eq. (4) becomes

$$J^\mu(\boldsymbol{y}) = \sum_{\text{particles}} q \int \delta^{(4)}(\boldsymbol{x}(t) - \boldsymbol{y}) \frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t}\, \mathrm{d}t = \sum_{\text{particles}} q\delta^{(3)}(\boldsymbol{x}(t) - \boldsymbol{y}) \frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t}, \tag{6}$$

where we have integrated over t and now the δ-function is 3-dimensional (including the 3 spatial dimensions). For $\mu = 0$, we have $\mathrm{d}x^0/\mathrm{d}t = 1$, and this simplifies to

$$J^0(\boldsymbol{y}) = \sum_{\text{particles}} q\delta^{(3)}(\boldsymbol{x}(t) - \boldsymbol{y}). \tag{7}$$

This is the usual charge density. Similarly, the ith spatial component has $\mathrm{d}x^i/\mathrm{d}t = v^i$, and

$$J^i(\boldsymbol{y}) = \sum_{\text{particles}} q v^i \delta^{(3)}(\boldsymbol{x}(t) - \boldsymbol{y}). \tag{8}$$

(c) [6 points] *Now let's consider the variation δS of the action to first order when there is a small change δA_μ in the 4-vector potential. Show that*

$$\delta S = \int (J^\mu \delta A_\mu - F^{\gamma\mu}\partial_\gamma \delta A_\mu)\, \mathrm{d}^4\boldsymbol{x}. \tag{9}$$

Conclude, using integration by parts, that the equation of motion is

$$-F^{\gamma\mu}{}_{,\gamma} = J^\mu. \tag{10}$$

We may write the variation with respect to A_μ (leaving the particle trajectories fixed):

$$\begin{aligned}
\delta S &= \delta\left[\sum_{\text{particles}} -m\int \mathrm{d}\tau + \int\left(J^\mu A_\mu - \frac{1}{4}F_{\gamma\mu}F^{\gamma\mu}\,\mathrm{d}^4\boldsymbol{x}\right)\right] \\
&= \int\left(J^\mu \delta A_\mu - \frac{1}{4}F_{\gamma\mu}\delta F^{\gamma\mu} - \frac{1}{4}\delta F_{\gamma\mu}F^{\gamma\mu}\right)\mathrm{d}^4\boldsymbol{x} \\
&= \int\left(J^\mu \delta A_\mu - \frac{1}{2}\delta F_{\gamma\mu}F^{\gamma\mu}\right)\mathrm{d}^4\boldsymbol{x} \\
&= \int\left[J^\mu \delta A_\mu - \frac{1}{2}(\partial_\gamma \delta A_\mu - \partial_\mu \delta A_\gamma)F^{\gamma\mu}\right]\mathrm{d}^4\boldsymbol{x} \\
&= \int\left[J^\mu \delta A_\mu - (\partial_\gamma \delta A_\mu)F^{\gamma\mu}\right]\mathrm{d}^4\boldsymbol{x},
\end{aligned} \tag{11}$$

where in the last step we used the fact that $F^{\gamma\mu}$ is antisymmetric to combine the two terms.

Now using integration by parts, we see that

$$\int -(\partial_\gamma \delta A_\mu)F^{\gamma\mu}\,\mathrm{d}^4\boldsymbol{x} = \int \delta A_\mu \partial_\gamma F^{\gamma\mu}\,\mathrm{d}^4\boldsymbol{x} + [\text{boundary terms}], \tag{12}$$

so overall the variation of the action fixing the boundaries can be written:

$$\delta S = \int (J^\mu + \partial_\gamma F^{\gamma\mu})\,\delta A_\mu\,\mathrm{d}^4\boldsymbol{x}. \tag{13}$$

For this to be zero for any variation, we have

$$J^\mu + \partial_\gamma F^{\gamma\mu} = 0 \quad\Rightarrow\quad -F^{\gamma\mu}{}_{,\gamma} = J^\mu. \tag{14}$$

(d) [4 points] *By explicitly writing the components, show that the equations of motion from (c) correspond to Gauss's law and to Ampère's law (including displacement current).*

Equation (10) has 4 components. Let's look at the $\mu = 0$ component first; this says

$$-F^{\gamma 0}{}_{,\gamma} = J^0, \tag{15}$$

or – noting that $J^0 = \rho$ and that $F^{00} = 0$ and $F^{i0} = -F_{i0} = -E_i$:

$$\rho = J^0 = -F^{\gamma 0}{}_{,\gamma} = -F^{00}{}_{,0} - F^{i0}{}_{,i} = -F^{i0}{}_{,i} = E_{i,i} = \boldsymbol{\nabla}\cdot\boldsymbol{E}, \tag{16}$$

where the last expression is a 3-dimensional divergence. This is Gauss's law.

For the spatial components, we may recall from HW#2 that $F_{ij} = \varepsilon_{ijk}B_k$ and write

$$J^j = -F^{\gamma j}{}_{,\gamma} = -F^{0j}{}_{,0} - F^{ij}{}_{,i} = F_{0j,0} - F_{ij,i} = -E_{j,0} - \varepsilon_{ijk}B_{k,i} = -\dot{E}_j + \varepsilon_{jik}\partial_i B_k = (-\dot{\boldsymbol{E}} + \boldsymbol{\nabla}\times\boldsymbol{B})^j, \tag{17}$$

where again the last step is a 3-dimensional equation. This is part of the equation

$$\boldsymbol{J} = -\dot{\boldsymbol{E}} + \boldsymbol{\nabla} \times \boldsymbol{B}, \tag{18}$$

which is Ampère's law.

(e) [4 points] *Explain why the 3-form $\tilde{\boldsymbol{d}}\boldsymbol{F} = 0$. Show that the 4 independent components of this equation correspond to the divergencelessness of the magnetic field and to Faraday's law of induction.*

Since $\boldsymbol{F} = \tilde{\boldsymbol{d}}\boldsymbol{A}$, we have

$$\tilde{\boldsymbol{d}}\boldsymbol{F} = \tilde{\boldsymbol{d}}\tilde{\boldsymbol{d}}\boldsymbol{A} = 0 \tag{19}$$

(the second exterior derivative is zero).

We can understand this result by considering each of the $\binom{4}{3} = 4$ independent components. Let's first take the 123 component:

$$[\tilde{\boldsymbol{d}}\boldsymbol{F}]_{123} = 3\partial_{[1}F_{23]} = \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = \partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = \boldsymbol{\nabla} \cdot \boldsymbol{B}. \tag{20}$$

This component of $\tilde{\boldsymbol{d}}\boldsymbol{F} = 0$ is thus the usual $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$, which states that there are no magnetic monopoles.

Now let's look at the 012 component. This says

$$[\tilde{\boldsymbol{d}}\boldsymbol{F}]_{012} = \partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = \frac{\partial B_z}{\partial t} + \frac{\partial E_y}{\partial x} + \frac{\partial(-E_x)}{\partial y} = (\dot{\boldsymbol{B}} + \boldsymbol{\nabla} \times \boldsymbol{E})_z. \tag{21}$$

This is the z-component of Faraday's law:

$$\dot{\boldsymbol{B}} + \boldsymbol{\nabla} \times \boldsymbol{E} = 0. \tag{22}$$

(f) [4 points] *Show that the equation of motion from (c) can be written in terms of differential forms as:*

$$\tilde{\boldsymbol{d}}({}^\star\boldsymbol{F}) = -{}^\star\boldsymbol{J}. \tag{23}$$

What type of form is on both sides of this equation?

Let's start by figuring out the rank of the form in Eq. (23). Since $\boldsymbol{F}$ is a 2-form in 4 dimensions, ${}^\star\boldsymbol{F}$ is a $4-2=2$-form, and $\tilde{\boldsymbol{d}}({}^\star\boldsymbol{F})$ is a 3-form. Similarly, $\boldsymbol{J}$ is a vector or (with metric) 1-form, and ${}^\star\boldsymbol{J}$ is a $4-1=3$-form.

There are two approaches to proving this – one by exhibiting a particular component and giving a symmetry argument that the others hold as well, and the other by working with each specific index. We do both methods. The first is shorter and (in my view) it is easier to keep track of what you are doing.

Specific case method. This is an easier method; the idea is that of the 4 independent components of Eq. (23), we only need to prove one. If this result is true in any reference frame, then all 4 components of the equation must be correct. We choose the 123 component for simplicity.

The 123 component of the left-hand side of Eq. (23) is

$$\begin{aligned}
[\tilde{\boldsymbol{d}}({}^\star\boldsymbol{F})]_{123} &= \partial_1({}^\star\boldsymbol{F})_{23} + \partial_2({}^\star\boldsymbol{F})_{31} + \partial_3({}^\star\boldsymbol{F})_{12} \\
&= \partial_1(\varepsilon_{2301}F^{01}) + \partial_2(\varepsilon_{3102}F^{02}) + \partial_3(\varepsilon_{1203}F^{03}) \\
&= \partial_1 F^{01} + \partial_2 F^{02} + \partial_3 F^{03} \\
&= \partial_\mu F^{0\mu} = -F^{\mu 0}{}_{,\mu} = J^0.
\end{aligned} \tag{24}$$

(In the second line, we wrote down only the terms in the dual relation that are non-zero, and did not write terms that are equivalent by antisymmetry. In the fourth line, we used that $F^{00} = 0$ to write an index sum.) However, the right-hand side is

$$-[{}^\star\boldsymbol{J}]_{123} = -\varepsilon_{1230}J^0 = J^0. \tag{25}$$

These are equal.

Index manipulation method. In index notation, we are trying to prove Eq. (23) or:

$$\frac{3}{2}\partial_{[\alpha}(\varepsilon_{\beta\gamma]\delta\epsilon}F^{\delta\epsilon}) \stackrel{?}{=} -\varepsilon_{\alpha\beta\gamma\delta}J^\delta. \tag{26}$$

(The 3 is from the exterior derivative, and the 2 is from the definition of the dual.) Let's take the left-hand side. Note there is an implied sum over δ and ϵ. Further, the answer is automatically zero by antisymmetry unless α, β, and γ are all different; we will define ω to be the index that is **not** in $\{\alpha, \beta, \gamma\}$. The left-hand side is then:

$$\text{L.H.S.} = \frac{1}{2}\sum_{\delta\epsilon}\left[\partial_\alpha(\varepsilon_{\beta\gamma\delta\epsilon}F^{\delta\epsilon}) + \partial_\beta(\varepsilon_{\gamma\alpha\delta\epsilon}F^{\delta\epsilon}) + \partial_\gamma(\varepsilon_{\alpha\beta\delta\epsilon}F^{\delta\epsilon})\right] \quad \text{(sums explicit)}. \tag{27}$$

Consider the terms where both δ and ϵ are not equal to ω, then in each term, the Levi-Civita tensor must have a repeated index since none of the indices are zero. Thus these terms vanish. Similarly if $\delta = \epsilon$, then antisymmetry of $\boldsymbol{F}$ implies these terms vanish. We are left with the case where one of δ or ϵ is equal to ω and the other is not. Antisymmetry of $\boldsymbol{F}$ means that we will get the same contribution from both cases, so let's take one case ($\delta = \omega$ and $\epsilon \neq \omega$) and multiply by 2:

$$\text{L.H.S.} = \sum_{\epsilon\neq\omega}\left[\partial_\alpha(\varepsilon_{\beta\gamma\omega\epsilon}F^{\omega\epsilon}) + \partial_\beta(\varepsilon_{\gamma\alpha\omega\epsilon}F^{\omega\epsilon}) + \partial_\gamma(\varepsilon_{\alpha\beta\omega\epsilon}F^{\omega\epsilon})\right] \tag{28}$$

Then in each term, only one of the values of ϵ gives 4 distinct indices for $\boldsymbol{\varepsilon}$:

$$\text{L.H.S.} = \partial_\alpha(\varepsilon_{\beta\gamma\omega\alpha}F^{\omega\alpha}) + \partial_\beta(\varepsilon_{\gamma\alpha\omega\beta}F^{\omega\beta}) + \partial_\gamma(\varepsilon_{\alpha\beta\omega\gamma}F^{\omega\gamma}) \quad \text{(no summation)}. \tag{29}$$

Re-ordering of the indices gives

$$\text{L.H.S.} = -\varepsilon_{\alpha\beta\gamma\omega}(\partial_\alpha F^{\omega\alpha} + \partial_\beta F^{\omega\beta} + \partial_\gamma F^{\omega\gamma}) \quad \text{(no summation)}. \tag{30}$$

Since $F^{\omega\omega} = 0$, the quantity in parentheses is actually

$$\sum_\mu \partial_\mu F^{\omega\mu} = F^{\omega\mu}{}_{,\mu} = -F^{\mu\omega}{}_{,\mu} = J^\omega; \tag{31}$$

thus

$$\text{L.H.S.} = -\varepsilon_{\alpha\beta\gamma\omega}J^\omega \quad \text{(no summation)}; \tag{32}$$

but this is $-\varepsilon_{\alpha\beta\gamma\delta}J^\delta$ **with** summation over δ, since only the $\delta = \sigma$ term contributes.

(g) [3 points] *Show that the equation of motion in (f)* requires *that the 4-divergence of* $\boldsymbol{J}$ *be zero. This is known as "automatic conservation of the source," and will occur in GR as well. Express this equation both as a differential form and in index notation.*

Mathematically, the second exterior derivative of a form is zero, so

$$-\tilde{\boldsymbol{d}}({}^\star\boldsymbol{J}) = \tilde{\boldsymbol{d}}\tilde{\boldsymbol{d}}\boldsymbol{F} = 0. \tag{33}$$

However, we learned in class that in an even number of dimensions, the exterior derivative of the dual of a vector is its divergence (times the Levi-Civita tensor):

$$\tilde{\boldsymbol{d}}({}^\star\boldsymbol{v}) = -(\boldsymbol{\nabla}\cdot\boldsymbol{v})\boldsymbol{\varepsilon} \tag{34}$$

for any vector field $\boldsymbol{v}$. Thus $\boldsymbol{\nabla}\cdot\boldsymbol{J} = 0$.

One could of course show this with indices:

$$J^\mu{}_{,\mu} = (-F^{\gamma\mu}{}_{,\gamma})_{,\mu} = -F^{\gamma\mu}{}_{,\gamma\mu} = 0 \tag{35}$$

since $\boldsymbol{F}$ is antisymmetric in $\mu \leftrightarrow \gamma$ but the second partial derivative is symmetric in $\mu \leftrightarrow \gamma$.

(h) [4 points] *Show that the field part of the action (Eq. 1) can be written as*

$$S = \int_{\mathbb{R}^4} -\frac{1}{2}\boldsymbol{F}\wedge{}^\star\boldsymbol{F}. \tag{36}$$

where $\mathbb{R}^4$ *is the 4-dimensional region over which we do the integral.*

We see that the 4-form $\boldsymbol{F} \wedge {}^\star\boldsymbol{F}$ has components:

$$
\begin{aligned}
[\boldsymbol{F} \wedge {}^\star\boldsymbol{F}]_{0123} &= 6F_{[01}({}^\star\boldsymbol{F})_{23]} \\
&= F_{01}({}^\star\boldsymbol{F})_{23} + F_{02}({}^\star\boldsymbol{F})_{31} + F_{03}({}^\star\boldsymbol{F})_{12} + F_{23}({}^\star\boldsymbol{F})_{01} + F_{31}({}^\star\boldsymbol{F})_{02} + F_{12}({}^\star\boldsymbol{F})_{03} \\
&= F_{01}F^{01} + F_{02}F^{02} + F_{03}F^{03} + F_{23}F^{23} + F_{31}F^{31} + F_{12}F^{12} \\
&= \frac{1}{2}F_{\mu\nu}F^{\mu\nu}.
\end{aligned} \tag{37}
$$

(The $\frac{1}{2}$ comes from the fact that in $F_{\mu\nu}F^{\mu\nu}$, each term – e.g., $F_{01}F^{01}$ – appears twice by antisymmetry, in this case also as $F_{10}F^{10} = F_{01}F^{01}$.) Using full antisymmetry, we get

$$
\boldsymbol{F} \wedge {}^\star\boldsymbol{F} = \left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\right)\boldsymbol{\varepsilon}. \tag{38}
$$

Therefore, the field action can be written as

$$
S_{\text{field}} = \int_{\mathbb{R}^4} -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,\mathrm{d}^4\boldsymbol{x} = \int_{\mathbb{R}^4} \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)\boldsymbol{\epsilon} = \int_{\mathbb{R}^4} -\frac{1}{2}\boldsymbol{F} \wedge {}^\star\boldsymbol{F}. \tag{39}
$$