Physics 6820 – Homework 2 Solutions

1. Charged particle motion in special relativity. [32 points]
In this problem, we consider the action for a particle in special relativity:

$$S = -m \int d\tau + q \int A_\mu(\boldsymbol{x})\, \mathrm{d}x^\mu. \tag{1}$$

Here $\boldsymbol{A}$ is a 1-form, m is the mass of the particle, and q is its electric charge.

(a) [3 points] *Suppose that the particle travels along a parameterized trajectory through spacetime, $x^\mu(\sigma)$. Write the action in terms of x^μ and $\mathrm{d}x^\mu/\mathrm{d}\sigma$.*

We can convert all of the differentials into σ differentials by the usual substitution:

$$S = -m \int \frac{\mathrm{d}\tau}{\mathrm{d}\sigma}\, \mathrm{d}\sigma + q \int A_\mu(\boldsymbol{x})\, \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\, \mathrm{d}\sigma. \tag{2}$$

Then we know from class that $\mathrm{d}\tau/\mathrm{d}\sigma = \sqrt{-\eta_{\mu\nu}(\mathrm{d}x^\mu/\mathrm{d}\sigma)(\mathrm{d}x^\nu/\mathrm{d}\sigma)}$, so

$$S = -m \int \sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}}\, \mathrm{d}\sigma + q \int A_\mu(\boldsymbol{x})\, \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\, \mathrm{d}\sigma. \tag{3}$$

(b) [5 points] *Find the canonical momenta p_μ. Show that in this case,*

$$p_\mu = m u_\mu + q A_\mu. \tag{4}$$

That is, the canonical momentum (defined by the Lagrangian procedure) is different from the mechanical momentum $\boldsymbol{p} = m\boldsymbol{u}$ that we computed in class.

Let's take the function f that is the integrand in $S = \int f\, \mathrm{d}\sigma$:

$$f = -m\sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}} + qA_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}. \tag{5}$$

Then p_γ is the partial derivative of this with respect to $\mathrm{d}x^\gamma/\mathrm{d}\sigma$:

$$\begin{aligned}
p_\gamma &= \frac{\partial}{\partial(\mathrm{d}x^\gamma/\mathrm{d}\sigma)}\left[-m\sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}} + qA_\mu\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\right] \\
&= -m\frac{1}{2\sqrt{-\eta_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\sigma}}}\left(-\eta_{\mu\nu}\delta^\mu_\gamma\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} - \eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\delta^\nu_\gamma\right) + qA_\mu\delta^\mu_\gamma \\
&= m\frac{1}{2\mathrm{d}\tau/\mathrm{d}\sigma}\left(\eta_{\gamma\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} + \eta_{\mu\gamma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\right) + qA_\gamma \\
&= m\frac{1}{\mathrm{d}\tau/\mathrm{d}\sigma}\left(\eta_{\gamma\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}\right) + qA_\gamma \\
&= m\eta_{\gamma\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + qA_\gamma \\
&= mu_\gamma + qA_\gamma.
\end{aligned} \tag{6}$$

Eq. (4) follows with a change in index label.

(c) [3 points] *Complete the Euler-Lagrange equations and show that*

$$\frac{\mathrm{d}p_\gamma}{\mathrm{d}\sigma} = q\frac{\partial A_\mu}{\partial x^\gamma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}. \tag{7}$$

We need to find $\partial f/\partial x^\gamma$. Now the only place in Eq. (3) that x^γ appears is in the vector potential, so

$$\frac{\partial f}{\partial x^\gamma} = \frac{\partial}{\partial x^\gamma}\left[qA_\mu(\boldsymbol{x})\,\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\right] = q\frac{\partial A_\mu}{\partial x^\gamma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}. \tag{8}$$

The Euler-Lagrange equation tells us $\mathrm{d}p_\gamma/\mathrm{d}\sigma = \partial f/\partial x^\gamma$; the result follows.

(d) [6 points] *Using the rules of calculus, show that your answer to (c) can be re-written as*

$$m\frac{\mathrm{d}u_\gamma}{\mathrm{d}\tau} = qF_{\gamma\mu}u^\mu, \tag{9}$$

where we have defined the field strength tensor $\mathbf{F}$ *by*

$$F_{\gamma\mu} = \frac{\partial A_\mu}{\partial x^\gamma} - \frac{\partial A_\gamma}{\partial x^\mu}. \tag{10}$$

Substituting the answer from (b) into (c), we have

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}(mu_\gamma + qA_\gamma) = q\frac{\partial A_\mu}{\partial x^\gamma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}. \tag{11}$$

By multiplying by $\mathrm{d}\sigma/\mathrm{d}\tau$ on both sides, we can change the variable to τ:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(mu_\gamma + qA_\gamma) = q\frac{\partial A_\mu}{\partial x^\gamma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = q\frac{\partial A_\mu}{\partial x^\gamma}u^\mu. \tag{12}$$

Now the chain rule tells us that

$$\frac{\mathrm{d}A_\gamma}{\mathrm{d}\tau} = \frac{\partial A_\gamma}{\partial x^\mu}\mathrm{d}x^\mu\mathrm{d}\tau = \frac{\partial A_\gamma}{\partial x^\mu}u^\mu, \tag{13}$$

so

$$m\frac{\mathrm{d}u_\gamma}{\mathrm{d}\sigma} + q\frac{\partial A_\gamma}{\partial x^\mu}u^\mu = q\frac{\partial A_\mu}{\partial x^\gamma}u^\mu \quad\Rightarrow\quad m\frac{\mathrm{d}u_\gamma}{\mathrm{d}\sigma} = q\left(\frac{\partial A_\mu}{\partial x^\gamma} - \frac{\partial A_\gamma}{\partial x^\mu}\right)u^\mu. \tag{14}$$

The result follows.

(e) [5 points] *Show that* $\mathbf{F}$ *is antisymmetric, and that it has the correct transformation properties to be a tensor.*

We see that $\mathbf{F}$ is antisymmetric because in Eq. (10), if we switch the two indices γ and μ, the right-hand side gets a $-$ sign (the two terms switch).

We can test the transformation properties by going to the barred frame:

$$\begin{aligned}
F_{\bar\gamma\bar\mu} &= \frac{\partial A_{\bar\mu}}{\partial x^{\bar\gamma}} - \frac{\partial A_{\bar\gamma}}{\partial x^{\bar\mu}} \\
&= \frac{\partial x^\delta}{\partial x^{\bar\gamma}}\frac{\partial([\mathbf{\Lambda}^{-1}]^\nu{}_{\bar\mu}A_\nu)}{\partial x^\delta} - \frac{\partial x^\nu}{\partial x^{\bar\mu}}\frac{\partial([\mathbf{\Lambda}^{-1}]^\delta{}_{\bar\gamma}A_\delta)}{\partial x^\nu} \\
&= [\mathbf{\Lambda}^{-1}]^\delta{}_{\bar\gamma}[\mathbf{\Lambda}^{-1}]^\nu{}_{\bar\mu}\frac{\partial A_\nu}{\partial x^\delta} - [\mathbf{\Lambda}^{-1}]^\nu{}_{\bar\mu}[\mathbf{\Lambda}^{-1}]^\delta{}_{\bar\gamma}\frac{\partial A_\delta}{\partial x^\nu} \\
&= [\mathbf{\Lambda}^{-1}]^\delta{}_{\bar\gamma}[\mathbf{\Lambda}^{-1}]^\nu{}_{\bar\mu}\left(\frac{\partial A_\nu}{\partial x^\delta} - \frac{\partial A_\delta}{\partial x^\nu}\right) \\
&= [\mathbf{\Lambda}^{-1}]^\delta{}_{\bar\gamma}[\mathbf{\Lambda}^{-1}]^\nu{}_{\bar\mu}F_{\delta\nu},
\end{aligned} \tag{15}$$

which is the correct transformation law.

(f) [6 points] *By considering the acceleration of slow-moving particles, show that the components of* $\mathbf{F}$ *are in accordance with*

$$F_{\gamma\mu} \to \left(\begin{array}{c|ccc} 0 & -E_x & -E_y & -E_z \\ \hline E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{array}\right). \tag{16}$$

Let's consider particles moving at velocities small compared to the speed of light, and work to order v (so $\gamma \approx 1$). Then the the spatial components ($\gamma = i$) of Eq. (9) say:

$$m\frac{\mathrm{d}u_i}{\mathrm{d}\tau} = qF_{i\mu}u^\mu = qF_{i0}u^0 + qF_{ij}u^j \tag{17}$$

(recall j is summed over 1,2,3). Now with $u_i \approx v^i$, $u^0 \approx 1$, and $\mathrm{d}\tau \approx \mathrm{d}t$, we have

$$m\frac{\mathrm{d}v^i}{\mathrm{d}t} = qF_{i0} + qF_{ij}v^j. \tag{18}$$

The left-hand side is the usual force of Newtonian physics. For stationary particles, there is a force qE_i, where $E_i = F_{i0}$ can be identified as the electric field. The antisymmetry of $\boldsymbol{F}$ then tells us $F_{0i} = -E_i$.

For moving particles, there is an additional force $qF_{ij}v^j$, which we want to identify as the i component of the magnetic force $q(\boldsymbol{v} \times \boldsymbol{B})_i = q\varepsilon_{ijk}v^j B^k$ (where $\boldsymbol{\varepsilon}$ here is the 3D Levi-Civita tensor). That is,

$$q\varepsilon_{ijk}v^j B^k = qF_{ij}v^j \quad \Rightarrow \quad F_{ij} = \varepsilon_{ijk}B^k. \tag{19}$$

Using this on each of the space-space components of $\boldsymbol{F}$ then gives Eq. (16).

(g) [4 points] *Write the independent components of Eq. (10) corresponding to E_x and B_x. Do these correspond to your notions (from undergraduate class) of how fields are derived from a vector potential? What is the physical interpretation of the 4th component, A_0?*

We see that for the electric field

$$E_x = F_{01} = \frac{\partial A_0}{\partial x^1} - \frac{\partial A_1}{\partial x^0}, \tag{20}$$

which is the 1-component of the more general rule

$$\boldsymbol{E} = \boldsymbol{\nabla} A_0 - \dot{\boldsymbol{A}} \tag{21}$$

(written with 3D equations and potentials). This is our usual notion of an electric field if $-A_0 = A^0$ is the electric potential.

For the magnetic field,

$$B_x = F_{23} = \frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3}, \tag{22}$$

which is the 1-component of the more general rule

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{23}$$

This is of course the usual rule for the magnetic field in terms of the vector potential.