Physics 6820 – Homework 2

(Dated: Due: September 6, 2019)

1. Charged particle motion in special relativity.

In this problem, we consider the action for a particle in special relativity:

$$S = -m \int d\tau + q \int A_\mu(\boldsymbol{x})\, \mathrm{d}x^\mu. \tag{1}$$

Here $\boldsymbol{A}$ is a 1-form, m is the mass of the particle, and q is its electric charge.

(a) [3 points] Suppose that the particle travels along a parameterized trajectory through spacetime, $x^\mu(\sigma)$. Write the action in terms of x^μ and $\mathrm{d}x^\mu/\mathrm{d}\sigma$.

(b) [5 points] Find the canonical momenta p_μ. Show that in this case,

$$p_\mu = m u_\mu + q A_\mu. \tag{2}$$

That is, the canonical momentum (defined by the Lagrangian procedure) is different from the mechanical momentum $\boldsymbol{p} = m\boldsymbol{u}$ that we computed in class.

(c) [3 points] Complete the Euler-Lagrange equations and show that

$$\frac{\mathrm{d}p_\gamma}{\mathrm{d}\sigma} = q \frac{\partial A_\mu}{\partial x^\gamma} \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}. \tag{3}$$

(d) [6 points] Using the rules of calculus, show that your answer to (c) can be re-written as

$$m \frac{\mathrm{d}u_\gamma}{\mathrm{d}\tau} = q F_{\gamma\mu} u^\mu, \tag{4}$$

where we have defined the *field strength tensor* $\mathbf{F}$ by

$$F_{\gamma\mu} = \frac{\partial A_\mu}{\partial x^\gamma} - \frac{\partial A_\gamma}{\partial x^\mu}. \tag{5}$$

(e) [5 points] Show that $\mathbf{F}$ is antisymmetric, and that it has the correct transformation properties to be a tensor.

(f) [6 points] By considering the acceleration of slow-moving particles, show that the components of $\mathbf{F}$ are in accordance with

$$F_{\gamma\mu} \to \left(\begin{array}{c|ccc} 0 & -E_x & -E_y & -E_z \\ \hline E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{array} \right). \tag{6}$$

(g) [4 points] Write the independent components of Eq. (5) corresponding to E_x and B_x. Do these correspond to your notions (from undergraduate class) of how fields are derived from a vector potential? What is the physical interpretation of the 4th component, A_0?