Physics 6820 – Homework 1 Solutions

1. The sky as viewed from a spaceship. [15 points]

Let's suppose that observer $\mathcal{O}$ remains on Earth (in the lab frame). Observer $\bar{\mathcal{O}}$ moves in a spaceship at velocity $V = \tanh\alpha$ in the z-direction with respect to Earth. As you may recall from watching science-fiction movies, if V is large enough, $\bar{\mathcal{O}}$ sees the stars appear to bunch up in front of them (the $+z$ direction). This problem works through the effect.

We suppose that the direction to the star makes an angle θ to the z-axis as seen from Earth, and $\bar{\theta}$ as seen from the spaceship. Without loss of generality, we will place the direction to the star at zero longitude (i.e., in the xz-plane).

(a) [2 points] *Show that in the Earth's frame, in time Δt, a photon from the star undergoes a displacement $\Delta x^\alpha = (\Delta t, -\Delta t\,\sin\theta, 0, -\Delta t\,\cos\theta)$.*

The photon's time displacement is $\Delta x^0 = \Delta t$ by definition. Since it travels at the speed of light, the magnitude of its spatial displacement is Δt. The direction stated in the problem corresponds to the unit vector $(\sin\theta, 0, \cos\theta)$, but that is the direction the photon is from, so there is a $-$ sign. The result follows.

(b) [5 points] *Apply a Lorentz transformation to find the photon's displacement in $\bar{\mathcal{O}}$'s frame. You may leave some results in terms of $\gamma = 1/\sqrt{1-V^2}$. Show that in the barred frame, the direction of the photon satisfies*

$$\cos\bar{\theta} = \frac{V + \cos\theta}{1 + V\cos\theta}. \tag{1}$$

The photon's displacement in the barred frame is

$$\Delta\bar{\boldsymbol{x}} = \left(\begin{array}{c|ccc} \gamma & 0 & 0 & -V\gamma \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -V\gamma & 0 & 0 & \gamma \end{array}\right) \begin{pmatrix} \Delta t \\ -\Delta t\,\sin\theta \\ 0 \\ -\Delta t\,\cos\theta \end{pmatrix} = \begin{pmatrix} \gamma(1+V\cos\theta)\Delta t \\ -\sin\theta\,\Delta t \\ 0 \\ -\gamma(V+\cos\theta)\Delta t \end{pmatrix}. \tag{2}$$

Now then $\cos\bar{\theta}$ is the $\bar{z}$-displacement of the photon $\Delta x^{\bar{3}}$ divided by the spatial displacement in the barred frame (which is $\Delta x^{\bar{0}}$ because the photon travels at the speed of light in the barred frame), with an extra $-$ sign because we want the direction to the star instead of the direction the photon is going:

$$\cos\bar{\theta} = \frac{-\Delta x^{\bar{3}}}{\Delta x^{\bar{0}}} = \frac{\gamma(V+\cos\theta)\Delta t}{\gamma(1+V\cos\theta)\Delta t} = \frac{V+\cos\theta}{1+V\cos\theta}. \tag{3}$$

Note — There is an alternate solution where you write the denominator in Eq. (3) as the magnitude of the spatial displacement, and then use trigonometric identities to simplify:

$$\begin{aligned} \sqrt{(\Delta x^{\bar{1}})^2 + (\Delta x^{\bar{2}})^2 + (\Delta x^{\bar{3}})^2} &= \sqrt{(-\sin\theta\,\Delta t)^2 + 0^2 + [-\gamma(V+\cos\theta)\Delta t]^2} \\ &= \sqrt{\sin^2\theta + \gamma^2(V+\cos\theta)^2}\,\Delta t \\ &= \sqrt{\gamma^{-2}\sin^2\theta + (V+\cos\theta)^2}\,\gamma\Delta t \\ &= \sqrt{(1-V^2)\sin^2\theta + V^2 + 2V\cos\theta + \cos^2\theta}\,\gamma\Delta t \\ &= \sqrt{(1+V^2\cos^2\theta + 2V\cos\theta}\,\gamma\Delta t \\ &= (1+V\cos\theta)\gamma\Delta t, \end{aligned} \tag{4}$$

which gives the same answer as using $\Delta x^{\bar{0}}$.

(c) [3 points] *Show that a star that appears on the "Equator" as seen from Earth ($\theta = \pi/2$) has an apparent position $\bar{\theta} = \cos^{-1} V$ as seen from the spaceship. How far from the North Pole does the star appear in the spaceship frame if $V = 0.9c$? What about $0.99c$?*

If we plug $\theta = \pi/2$ or $\cos\theta = 0$ into Eq. (1), then we get $\cos\bar{\theta} = V$. This implies $\bar{\theta} = \cos^{-1} V$. Since $|V| < 1$, this is always a real number.

The angle from the North Celestial Pole to the apparent position of the star in the spaceship frame is $\bar{\theta}$. By taking the inverse cosine, we see that this is 25.8° at $V = 0.9c$ and 8.1° at $V = 0.99c$.

(d) [5 points] *Now take the limit of small $\theta \ll 1$ (i.e., we will consider a constellation that contains the North Pole). Show that*

$$\bar\theta \approx \sqrt{\frac{1-V}{1+V}}\,\theta. \tag{5}$$

[Hint: Take the Taylor expansion of your answer to (a) to 2nd order in θ.] This means that the constellation containing the North Pole appears shrunk by a factor of $\sqrt{(1-V)/(1+V)}$ when seen from the spaceship.

Let's take the Taylor expansion of Eq. (**??**), keeping terms through order θ^2:

$$\begin{aligned}\cos\bar\theta &= \frac{V+\cos\theta}{1+V\cos\theta}\\ &= \frac{V+1-\frac{1}{2}\theta^2}{1+V-\frac{1}{2}V\theta^2} + \mathcal{O}(\theta^4)\\ &= \frac{V+1-\frac{1}{2}V\theta^2-\frac{1}{2}(1-V)\theta^2}{1+V-\frac{1}{2}V\theta^2} + \mathcal{O}(\theta^4)\\ &= 1-\frac{\frac{1}{2}(1-V)\theta^2}{1+V-\frac{1}{2}V\theta^2} + \mathcal{O}(\theta^4)\\ &= 1-\frac{\frac{1}{2}(1-V)\theta^2}{1+V} + \mathcal{O}(\theta^4).\end{aligned} \tag{6}$$

(In the last step, since the numerator is only nonzero to order θ^2, we can drop higher-order terms in the denominator.) We can set this equal to the Taylor expansion of the left-hand side:

$$1-\frac{1}{2}\bar\theta^2 = 1-\frac{\frac{1}{2}(1-V)\theta^2}{1+V} + [4^{\rm th}\,\text{order terms}], \tag{7}$$

which simplifies to Eq. (5).

Note — There is an alternate solution where one works with $\sin\bar\theta$ (and thus can do 1st order Taylor expansions). This is

$$\sin\bar\theta = \frac{-\Delta x^{\bar 1}}{\Delta x^{\bar 0}} = \frac{\sin\theta\,\Delta t}{\gamma(V+\cos\theta)\Delta t} = \frac{\sin\theta}{\gamma(V+\cos\theta)}; \tag{8}$$

to 1st order in θ, we can now set $\cos\theta \approx 1$ in the denominator, and $\sin\theta\approx\theta$ and $\sin\bar\theta\approx\bar\theta$. Then:

$$\bar\theta \approx \frac{\theta}{\gamma(V+1)} = \frac{\sqrt{1-V^2}}{V+1}\,\theta = \sqrt{\frac{1-V^2}{(1+V)^2}}\,\theta = \sqrt{\frac{1-V}{1+V}}\,\theta. \tag{9}$$

2. The "law of cosines" for addition of velocities. [17 points]

In this problem, we will consider three observers: $\mathcal{O}_{\rm A}$, $\mathcal{O}_{\rm B}$, and $\mathcal{O}_{\rm C}$. In the frame of $\mathcal{O}_A$, observer $\mathcal{O}_{\rm B}$ has velocity $\boldsymbol{V}_{\rm AB}$, and $\mathcal{O}_{\rm C}$ has velocity $\boldsymbol{V}_{\rm AC}$. The 3D angle between $\boldsymbol{V}_{\rm AB}$ and $\boldsymbol{V}_{\rm AC}$ as seen by $\mathcal{O}_{\rm A}$ is denoted $\theta_{\rm BAC}$. Our objective is to determine from this information the magnitude of the velocity of $\mathcal{O}_{\rm C}$ in the frame of $\mathcal{O}_{\rm B}$ ($V_{\rm BC}$).

In this problem, you will probably want to write $\gamma_{\rm AB} = 1/\sqrt{1-V_{\rm AB}^2}$, etc.

Without loss of generality, you may place $\boldsymbol{V}_{\rm AB}$ on the z-axis, and $\boldsymbol{V}_{\rm AC}$ in the xz-plane.

(a) [3 points] *Explain why in $\mathcal{O}_{\rm A}$'s frame, the world line of $\mathcal{O}_{\rm C}$ can be written as*

$$x_{\rm C\,in\,A} = V_{\rm AC}\sin\theta_{\rm BAC}\,t_{\rm C\,in\,A},\quad y_{\rm C\,in\,A}=0,\quad\text{and}\quad z_{\rm C\,in\,A} = V_{\rm AC}\cos\theta_{\rm BAC}\,t_{\rm C\,in\,A}. \tag{10}$$

The velocity of $\mathcal{O}_{\rm C}$ in $\mathcal{O}_{\rm A}$'s frame is $(V_{\rm AC}\sin\theta_{\rm BAC}, 0, V_{\rm AC}\cos\theta_{\rm BAC})$. Multiplying the velocity by the time gives the position of $\mathcal{O}_{\rm C}$ shown in Eq. (10).

(b) [5 points] *Now do the Lorentz transform to $\mathcal{O}_{\rm B}$'s frame, and show that*

$$x_{\rm C\,in\,B} = \frac{V_{\rm AC}\sin\theta_{\rm BAC}}{\gamma_{\rm AB}(1-V_{\rm AB}V_{\rm AC}\cos\theta_{\rm BAC})}t_{\rm C\,in\,B},\quad y_{\rm C\,in\,B}=0,\quad\text{and}\quad z_{\rm C\,in\,B} = \frac{V_{\rm AC}\cos\theta_{\rm BAC}-V_{\rm AB}}{1-V_{\rm AB}V_{\rm AC}\cos\theta_{\rm BAC}}t_{\rm C\,in\,B}. \tag{11}$$

We do the Lorentz transformation,

$$\begin{pmatrix} t_{\mathrm{C\,in\,B}} \\ x_{\mathrm{C\,in\,B}} \\ y_{\mathrm{C\,in\,B}} \\ z_{\mathrm{C\,in\,B}} \end{pmatrix} = \mathbf{\Lambda}_{\mathcal{O}_\mathrm{A}\to\mathcal{O}_\mathrm{B}} \begin{pmatrix} t_{\mathrm{C\,in\,A}} \\ x_{\mathrm{C\,in\,A}} \\ y_{\mathrm{C\,in\,A}} \\ z_{\mathrm{C\,in\,A}} \end{pmatrix} = \left(\begin{array}{c|ccc} \gamma_{\mathrm{AB}} & 0 & 0 & -V_{\mathrm{AB}}\gamma_{\mathrm{AB}} \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -V_{\mathrm{AB}}\gamma_{\mathrm{AB}} & 0 & 0 & \gamma_{\mathrm{AB}} \end{array}\right) \begin{pmatrix} t_{\mathrm{C\,in\,A}} \\ V_{\mathrm{AC}}\sin\theta_{\mathrm{BAC}}\, t_{\mathrm{C\,in\,A}} \\ 0 \\ V_{\mathrm{AC}}\cos\theta_{\mathrm{BAC}}\, t_{\mathrm{C\,in\,A}} \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_{\mathrm{AB}} t_{\mathrm{C\,in\,A}}(1 - V_{\mathrm{AB}}V_{\mathrm{AC}}\cos\theta_{\mathrm{BAC}}) \\ V_{\mathrm{AC}}\sin\theta_{\mathrm{BAC}}\, t_{\mathrm{C\,in\,A}} \\ 0 \\ \gamma_{\mathrm{AB}} t_{\mathrm{C\,in\,A}}(V_{\mathrm{AC}}\cos\theta_{\mathrm{BAC}} - V_{\mathrm{AB}}) \end{pmatrix}. \tag{12}$$

The result follows.

(c) [5 points] *Now show that the squared velocity of $\mathcal{O}_\mathrm{C}$ in $\mathcal{O}_\mathrm{B}$'s frame is*

$$V_{\mathrm{BC}}^2 = \frac{V_{\mathrm{AC}}^2 + V_{\mathrm{AB}}^2 - 2V_{\mathrm{AB}}V_{\mathrm{AC}}\cos\theta_{\mathrm{BAC}} - V_{\mathrm{AB}}^2 V_{\mathrm{AC}}^2 \sin^2\theta_{\mathrm{BAC}}}{(1 - V_{\mathrm{AB}}V_{\mathrm{AC}}\cos\theta_{\mathrm{BAC}})^2}. \tag{13}$$

The components of the velocity of $\mathcal{O}_\mathrm{C}$ in $\mathcal{O}_\mathrm{B}$'s frame can be read off from Eq. (11). Then V_{BC}^2 is the sum of their squares:

$$\begin{aligned} V_{\mathrm{BC}}^2 &= \left[\frac{V_{\mathrm{AC}}\sin\theta_{\mathrm{BAC}}}{\gamma_{\mathrm{AB}}(1 - V_{\mathrm{AB}}V_{\mathrm{AC}}\cos\theta_{\mathrm{BAC}})}\right]^2 + 0^2 + \left[\frac{V_{\mathrm{AC}}\cos\theta_{\mathrm{BAC}} - V_{\mathrm{AB}}}{1 - V_{\mathrm{AB}}V_{\mathrm{AC}}\cos\theta_{\mathrm{BAC}}}\right]^2 \\ &= \frac{(V_{\mathrm{AC}}\sin\theta_{\mathrm{BAC}})^2(1 - V_{\mathrm{AB}}^2) + (V_{\mathrm{AC}}\cos\theta_{\mathrm{BAC}} - V_{\mathrm{AB}})^2}{(1 - V_{\mathrm{AB}}V_{\mathrm{AC}}\cos\theta_{\mathrm{BAC}})^2} \\ &= \frac{V_{\mathrm{AC}}^2\sin^2\theta_{\mathrm{BAC}} - V_{\mathrm{AC}}^2V_{\mathrm{AB}}^2\sin^2\theta_{\mathrm{BAC}} + V_{\mathrm{AC}}^2\cos^2\theta_{\mathrm{BAC}} - 2V_{\mathrm{AB}}V_{\mathrm{AC}}\cos\theta_{\mathrm{BAC}} + V_{\mathrm{AB}}^2}{(1 - V_{\mathrm{AB}}V_{\mathrm{AC}}\cos\theta_{\mathrm{BAC}})^2} \\ &= \frac{V_{\mathrm{AB}}^2 + V_{\mathrm{AC}}^2 - 2V_{\mathrm{AB}}V_{\mathrm{AC}}\cos\theta_{\mathrm{BAC}} - V_{\mathrm{AB}}^2V_{\mathrm{AC}}^2\sin^2\theta_{\mathrm{BAC}}}{(1 - V_{\mathrm{AB}}V_{\mathrm{AC}}\cos\theta_{\mathrm{BAC}})^2}. \end{aligned} \tag{14}$$

(d) [2 points] *Show that this reduces to the usual law of cosines for small velocities. (This should not involve any messy algebra; you should be able to do it in one line with some explanation.)*

If we drop terms of order V^4 in Eq. (13), then the $V_{\mathrm{AB}}^2V_{\mathrm{AC}}^2\sin^2\theta_{\mathrm{BAC}}$ disappears from the numerator. Since the numerator is already of order V^2, and the denominator is of order 1, then the $-V_{\mathrm{AB}}V_{\mathrm{AC}}\cos\theta_{\mathrm{BAC}}$ term can be dropped from the denominator. This leads to

$$V_{\mathrm{BC}}^2 = V_{\mathrm{AB}}^2 + V_{\mathrm{AC}}^2 - 2V_{\mathrm{AB}}V_{\mathrm{AC}}\cos\theta_{\mathrm{BAC}}, \tag{15}$$

which is the standard law of cosines – it is how velocities add in Newtonian physics.

(e) [2 points] *What is the relative velocity of two spaceships that are both moving at $0.9c$ relative to the Earth, but in perpendicular directions?*

For perpendicular directions, $\cos\theta_{\mathrm{BAC}} = 0$ and $\sin\theta_{\mathrm{BAC}} = 1$. This means that

$$V_{\mathrm{BC}}^2 = V_{\mathrm{AB}}^2 + V_{\mathrm{AC}}^2 - V_{\mathrm{AB}}^2V_{\mathrm{AC}}^2 \tag{16}$$

and then

$$V_{\mathrm{BC}}^2 = \sqrt{V_{\mathrm{AB}}^2 + V_{\mathrm{AC}}^2 - V_{\mathrm{AB}}^2V_{\mathrm{AC}}^2}. \tag{17}$$

For $V_{\mathrm{AB}} = V_{\mathrm{AC}} = 0.9c$, we find $V_{\mathrm{BC}} = 0.982c$.