Physics 6820 – Homework 1

(Dated: Due: August 30, 2019)

1. The sky as viewed from a spaceship. [15 points]

Let's suppose that observer $\mathcal{O}$ remains on Earth (in the lab frame). Observer $\bar{\mathcal{O}}$ moves in a spaceship at velocity $V = \tanh\alpha$ in the z-direction with respect to Earth. As you may recall from watching science-fiction movies, if V is large enough, $\bar{\mathcal{O}}$ sees the stars appear to bunch up in front of them (the $+z$ direction). This problem works through the effect.

We suppose that the direction to the star makes an angle θ to the z-axis as seen from Earth, and $\bar{\theta}$ as seen from the spaceship. Without loss of generality, we will place the direction to the star at zero longitude (i.e., in the xz-plane).

(a) [2 points] Show that in the Earth's frame, in time Δt, a photon from the star undergoes a displacement $\Delta x^\alpha = (\Delta t, -\Delta t\,\sin\theta, 0, -\Delta t\,\cos\theta)$.

(b) [5 points] Apply a Lorentz transformation to find the photon's displacement in $\bar{\mathcal{O}}$'s frame. You may leave some results in terms of $\gamma = 1/\sqrt{1-V^2}$. Show that in the barred frame, the direction of the photon satisfies

$$\cos\bar{\theta} = \frac{V+\cos\theta}{1+V\cos\theta}. \tag{1}$$

(c) [3 points] Show that a star that appears on the "Equator" as seen from Earth ($\theta = \pi/2$) has an apparent position $\bar{\theta} = \cos^{-1} V$ as seen from the spaceship. How far from the North Pole does the star appear in the spaceship frame if $V = 0.9c$? What about $0.99c$?

(d) [5 points] Now take the limit of small $\theta \ll 1$ (i.e., we will consider a constellation that contains the North Pole). Show that

$$\bar{\theta} \approx \sqrt{\frac{1-V}{1+V}}\,\theta. \tag{2}$$

[*Hint*: Take the Taylor expansion of your answer to (a) to 2nd order in θ.] This means that the constellation containing the North Pole appears shrunk by a factor of $\sqrt{(1-V)/(1+V)}$ when seen from the spaceship.

2. The "law of cosines" for addition of velocities. [17 points]

In this problem, we will consider three observers: $\mathcal{O}_{\rm A}$, $\mathcal{O}_{\rm B}$, and $\mathcal{O}_{\rm C}$. In the frame of $\mathcal{O}_A$, observer $\mathcal{O}_{\rm B}$ has velocity $\boldsymbol{V}_{\rm AB}$, and $\mathcal{O}_{\rm C}$ has velocity $\boldsymbol{V}_{\rm AC}$. The 3D angle between $\boldsymbol{V}_{\rm AB}$ and $\boldsymbol{V}_{\rm AC}$ as seen by $\mathcal{O}_{\rm A}$ is denoted $\theta_{\rm BAC}$. Our objective is to determine from this information the magnitude of the velocity of $\mathcal{O}_{\rm C}$ in the frame of $\mathcal{O}_{\rm B}$ ($V_{\rm BC}$).

In this problem, you will probably want to write $\gamma_{\rm AB} = 1/\sqrt{1-V_{\rm AB}^2}$, etc.

Without loss of generality, you may place $\boldsymbol{V}_{\rm AB}$ on the z-axis, and $\boldsymbol{V}_{\rm AC}$ in the xz-plane.

(a) [3 points] Explain why in $\mathcal{O}_{\rm A}$'s frame, the world line of $\mathcal{O}_{\rm C}$ can be written as

$$x_{\rm C\,in\,A} = V_{\rm AC}\sin\theta_{\rm BAC}\,t_{\rm C\,in\,A}, \quad y_{\rm C\,in\,A} = 0, \quad \text{and} \quad z_{\rm C\,in\,A} = V_{\rm AC}\cos\theta_{\rm BAC}\,t_{\rm C\,in\,A}. \tag{3}$$

(b) [5 points] Now do the Lorentz transform to $\mathcal{O}_{\rm B}$'s frame, and show that

$$x_{\rm C\,in\,B} = \frac{V_{\rm AC}\sin\theta_{\rm BAC}}{\gamma_{\rm AB}(1-V_{\rm AB}V_{\rm AC}\cos\theta_{\rm BAC})}t_{\rm C\,in\,B}, \quad y_{\rm C\,in\,B} = 0, \quad \text{and} \quad z_{\rm C\,in\,B} = \frac{V_{\rm AC}\cos\theta_{\rm BAC} - V_{\rm AB}}{1-V_{\rm AB}V_{\rm AC}\cos\theta_{\rm BAC}}t_{\rm C\,in\,B}. \tag{4}$$

(c) [5 points] Now show that the squared velocity of $\mathcal{O}_{\rm C}$ in $\mathcal{O}_{\rm B}$'s frame is

$$V_{\rm BC}^2 = \frac{V_{\rm AC}^2 + V_{\rm AB}^2 - 2V_{\rm AB}V_{\rm AC}\cos\theta_{\rm BAC} - V_{\rm AB}^2V_{\rm AC}^2\sin^2\theta_{\rm BAC}}{(1-V_{\rm AB}V_{\rm AC}\cos\theta_{\rm BAC})^2}. \tag{5}$$

(d) [2 points] Show that this reduces to the usual law of cosines for small velocities. (This should not involve any messy algebra; you should be able to do it in one line with some explanation.)

(e) [2 points] What is the relative velocity of two spaceships that are both moving at $0.9c$ relative to the Earth, but in perpendicular directions?