Physics 6820 – Final Exam

1. Black holes with a cosmological constant. [26 points]

In this problem, we will consider a Schwarzschild-like black hole but with a cosmological constant Λ. You may take the fundamental constants $c = G = \hbar = k_{\rm B} = 1$.

In the first parts of this problem (a–c), we will solve for the metric of a black hole with Λ.

Suppose that we have no matter present (other than Λ) and that we have a static, spherically symmetric spacetime. Let's treat the cosmological constant as a type of matter (i.e., by moving it to the right-hand side of Einstein's equations so that it contributes an effective stress-energy tensor $[\mathbf{T}_\Lambda]_{\mu\nu} = -\frac{\Lambda}{8\pi} g_{\mu\nu}$). The advantage of this is that we can then use formulas from regular GR (i.e., with matter but no Λ). In particular, the Tolman-Oppenheimer-Volkoff metric (see Lecture XVI) should describe this situation:

$$ds^2 = -e^{2\Phi(r)}\,dt^2 + \frac{dr^2}{1 - 2m(r)/r} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2). \tag{1}$$

(a) [4 points] *Show that the differential equations for Φ and m are:*

$$\Phi' = \frac{m - \frac{1}{2}\Lambda r^3}{r(r-2m)} \quad \text{and} \quad m' = \frac{1}{2}\Lambda r^2, \tag{2}$$

where the $'$ denotes a derivative with respect to r.

If we take the Einstein equations with Λ moved to the right-hand side:

$$G_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{\Lambda}{8\pi} g_{\mu\nu}\right), \tag{3}$$

then we identify $-\frac{\Lambda}{8\pi} g_{\mu\nu}$ as an effective stress-energy tensor for the cosmological constant. If we try to write this new term in the form of a stress-energy tensor,

$$-\frac{\Lambda}{8\pi} g_{\mu\nu} \stackrel{?}{=} (\rho_\Lambda + p_\Lambda) u_\mu u_\nu + p_\Lambda g_{\mu\nu}, \tag{4}$$

then we identify the coefficient of $g_{\mu\nu}$ as the pressure of the cosmological constant, $p_\Lambda = -\frac{\Lambda}{8\pi}$; and the coefficient of $u_\mu u_\nu$ as $\rho_\Lambda + p_\Lambda = 0$ or $\rho_\Lambda = \frac{\Lambda}{8\pi}$.

Now let's review the TOV equations from Lecture XVI, Eqs. (21, 19, 22):

$$\Phi' = \frac{m + 4\pi r^3 p}{r(r-2m)}, \quad m' = 4\pi r^2 \rho, \quad \text{and} \quad p' = -\Phi'(\rho + p). \tag{5}$$

Assuming no matter other than Λ, we write $p = -\frac{\Lambda}{8\pi}$ and $\rho = \frac{\Lambda}{8\pi}$. The last equation is trivially satisfied. The first two give Eq. (2).

(b) [6 points] *Show that the solutions to these equations are:*

$$\Phi = \frac{1}{2}\ln\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) + \Phi_0 \quad \text{and} \quad m = M + \frac{1}{6}\Lambda r^3, \tag{6}$$

where M and Φ_0 are constants of integration. (Hint: solve for m first.)

The m equation from Eq. (2) can be simply integrated:

$$m = \int m'\,dr = \int \frac{1}{2}\Lambda r^2\,dr = \frac{1}{6}\Lambda r^3 + M, \tag{7}$$

where M is a constant. Plugging this into the Φ' equation gives

$$\Phi' = \frac{m - \frac{1}{2}\Lambda r^3}{r(r-2m)} = \frac{M + \frac{1}{6}\Lambda r^3 - \frac{1}{2}\Lambda r^3}{r(r - 2M - \frac{1}{3}\Lambda r^3)} = \frac{M - \frac{1}{3}\Lambda r^3}{r(r - 2M - \frac{1}{3}\Lambda r^3)}. \tag{8}$$

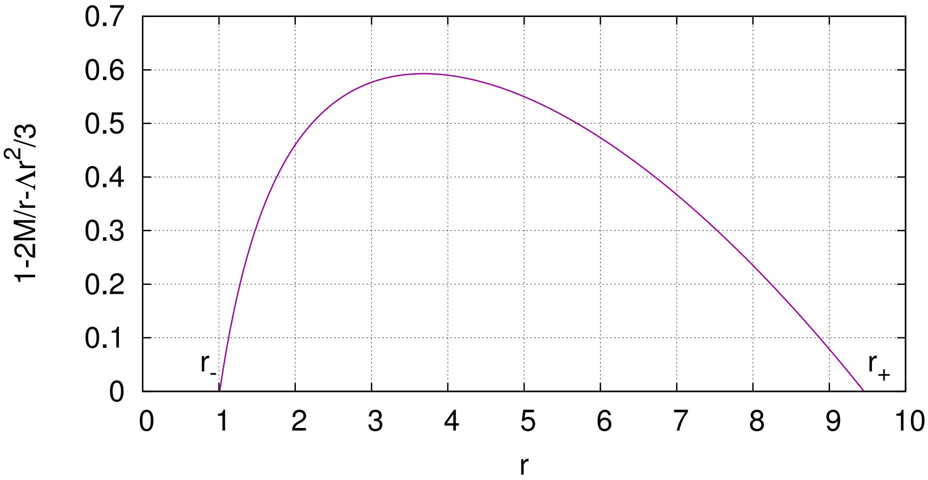

FIG. 1: A plot of the function $1 - 2M/r - \Lambda r^2/3$ for $M = 0.5$ and $\Lambda = 0.03$.

Then:

$$\Phi = \int \Phi' \,\mathrm{d}r = \int \frac{M - \frac{1}{3}\Lambda r^3}{r(r - 2M - \frac{1}{3}\Lambda r^3)} \,\mathrm{d}r = \frac{1}{2} \int \frac{2M/r^2 - \frac{2}{3}\Lambda r}{1 - 2M/r - \frac{1}{3}\Lambda r^2} \,\mathrm{d}r = \frac{1}{2} \ln \left(1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2\right) + \Phi_0, \tag{9}$$

where we used the fact that $2M/r^2 - \frac{2}{3}\Lambda r$ is the derivative of $1 - 2M/r - \frac{1}{3}\Lambda r^2$. In the last expression, Φ_0 is a constant of integration. This proves Eq. (6).

(c) [2 points] *Explain why Φ_0 can be set to zero without loss of generality. (I am looking for ~ 1 sentence.)*

We showed in Lecture XVI that in the TOV solution, re-scaling time ($t = \mathrm{e}^{\Delta}\bar{t}$) leads to a re-scaling of $\Phi(r) \to \Phi(r) + \Delta$. Thus the additive constant in Φ is irrelevant, and we may set $\Phi_0 = 0$.

The metric can thus be written in the form:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - 2M/r - \Lambda r^2/3} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \,\mathrm{d}\phi^2). \tag{10}$$

If $M \ll \Lambda^{-1/2}$, this metric is well-behaved in the regime $r_- < r < r_+$, where r_- and r_+ are the roots of the equation $1 - 2M/r - \Lambda r^2/3 = 0$.

(d) [5 points] *Again in the case $M \ll \Lambda^{-1/2}$, make a qualitative sketch of the function $1 - 2M/r - \Lambda r^2/3$, identifying the two roots. What is the physical significance of M, and which root looks like a Schwarzschild-type horizon and which like a de Sitter (cosmological) horizon? (I am looking for a plot and 1–2 sentences.)*

See Figure 1. At the inner root, the Λ term in $1 - 2M/r - \Lambda r^2/3$ is negligible and $r_- \approx 2M$; this looks like a Schwarzschild black hole of mass M. At the outer root, the M term is negligible and $r_+ \approx \sqrt{3/\Lambda}$; this looks like a de Sitter horizon.

I now want to solve for the area of the outer horizon for small M. Since r_+ is the root of a cubic equation, there is formally a solution with a bunch of cube roots, but I will use the Taylor expansion method below because it is faster computationally.

(e) [5 points] *Let's now suppose that the outer horizon has a radius with*

$$r_+^2 = c_0 + c_1 M + \ldots \quad \text{or} \quad r_+ = \sqrt{c_0 + c_1 M + \ldots}\,, \tag{11}$$

where c_0 and c_1 are the leading Taylor coefficients. By plugging this into $1 - 2M/r - \Lambda r^2/3 = 0$, and working to first *order in M, show that*

$$c_0 = \frac{3}{\Lambda} \quad \text{and} \quad c_1 = -2\sqrt{\frac{3}{\Lambda}}. \tag{12}$$

(This will be faster if you drop higher-order terms in M as quickly as you can.)

Let's plug Eq. (11) into $1 - 2M/r - \Lambda r^2/3 = 0$:

$$1 - 2M(c_0 + c_1 M + ...)^{-1/2} - \frac{1}{3}\Lambda(c_0 + c_1 M + ...) = 0. \tag{13}$$

Expanding the left-hand side to order M, we find

$$1 - \frac{1}{3}\Lambda c_0 + \left(-2c_0^{-1/2} - \frac{1}{3}\Lambda c_1\right) M + ... = 0. \tag{14}$$

The constant term gives $1 - \frac{1}{3}\Lambda c_0 = 0$, or $c_0 = 3/\Lambda$. The order-M term gives $-2c_0^{-1/2} - \frac{1}{3}\Lambda c_1 = 0$, or

$$c_1 = -6c_0^{-1/2}\Lambda^{-1} = -6\sqrt{\frac{\Lambda}{3}}\,\Lambda^{-1} = -2\sqrt{\frac{3}{\Lambda}}. \tag{15}$$

(f) [4 points] *If we identify the entropy S_+ of the outer horizon with $\frac{1}{4}$ of its area, show that*

$$\frac{\mathrm{d}S_+}{\mathrm{d}M} = -2\pi\sqrt{\frac{3}{\Lambda}}. \tag{16}$$

Compare this to the de Sitter temperature that we derived in class, $T_{\mathrm{deS}} = \frac{1}{2\pi}\sqrt{\frac{\Lambda}{3}}$. Does this make sense? What does the $-$ sign represent?

Now the outer horizon has entropy

$$S_+ = \frac{1}{4}A_+ = \frac{1}{4} \times 4\pi r_+^2 = \pi r_+^2 = \pi(c_0 + c_1 M + ...), \tag{17}$$

so

$$\frac{\mathrm{d}S_+}{\mathrm{d}M} = \pi c_1 = -2\pi\sqrt{\frac{3}{\Lambda}}. \tag{18}$$

This is $-1/T_{\mathrm{deS}}$. Normally we say that for a thermodynamic system with only energy as the conserved quantity (so we don't have to worry about partial derivatives), $\mathrm{d}S/\mathrm{d}E = 1/T$. So this looks just like the usual relation, but with a $-$ sign. The extra $-$ sign occurs because when an amount of energy ΔE is taken from small r and sent to the de Sitter horizon, the mass M at small r changes by $-\Delta E$. Thus the relation between horizon entropy and mass near the origin contains this $-$ sign.